During 16^{th} - 17^{th} centuries, finding solutions to polynomial equations $\rightarrow$ sport.

quadratics: easy... since biblical times.
cubics: tricks... no consistent solution
quartics: more tricks

People didn't know about imaginaries.. or Gauss's Fundamental Theorem: n^{th} degree polynomial $\Rightarrow$ n roots.

quintic: notsomuch.

Lagrange had the clue having to do with understanding the symmetries of the solutions. What do I mean?

Consider

$$x^2 - 6x + 4 = 0$$

call the 2 roots $\alpha, \beta = \dfrac{6 \pm \sqrt{20}}{2}$

$$= 3 \pm \sqrt{5}$$

$\alpha = 3 + \sqrt{5}$
$\beta = 3 - \sqrt{5}$

notice $\left.\begin{array}{l} \alpha + \beta = 6 \\ \alpha\beta = 4 \end{array}\right\}$ if we were to do $\begin{array}{l} \alpha \rightarrow \beta \\ \beta \rightarrow \alpha \end{array}$, $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$

These two equations would not change.

In fact, the Galois group of this polynomial is S_2

$\begin{pmatrix} \alpha & \beta \\ \alpha & \beta \end{pmatrix}$ & $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ for polynomial degree $d=2$ eh

How about $x^4 + 4x^2 - 5 = 0$

Lagrange would have found a way to reduce this to a cubic -- but without a real formal basis.

Investigate the symmetries here, $d = 4$

It factorizes: $(x^2+1)(x^2-5) = 0$

$\Rightarrow \quad \alpha = i$

$\beta = -i$

$\gamma = \sqrt{5}$

$\delta = -\sqrt{5}$

There are many equations like

$\alpha^2 + 1 = 0$ in which $\alpha \to \beta$ leave it invariant.

$\alpha + \beta = 0 \qquad \delta^2 - 5 = 0$

call the whole set K, which is ∞

There is also a set of equations like

$\alpha + \gamma = 0 \quad \notin K$ and are <u>false</u>
aren't invariant
wrt substitutions

also ∞ #.

Clearly, α & β and γ & δ are related only to one another. So, the set of substitutions is

$$R = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \gamma & \delta \end{pmatrix}$$

$$S = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \delta & \gamma \end{pmatrix}$$

$$T = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \end{pmatrix}$$

$$E = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$$

that's it for this equation

S_4 has $4! = 24$ elements.

The group of this equation

$$H = \{E, R, S, T\} \subseteq S_4$$

DEF. Index ; The index of a subgroup $G' \subseteq G$ division of the order of G, g, by the order g' of G'.. $i = \frac{g}{g'}$

In general, for the arbitrary quartic,

$$x^4 + px^2 + q = 0$$

solutions look like

$$\alpha = \sqrt{\frac{-p + \sqrt{p^2 - 4q}}{2}} \quad \text{and so on}$$

In general only $\underbrace{\alpha + \beta = 0 \quad \text{and} \quad \gamma + \delta = 0}_{K_1}$ hold.

$\alpha \to \beta \quad \beta \to \alpha$

$\alpha \to \gamma \quad \beta \to \delta$

$\vdots$

there are 8 $\longrightarrow$ $J = \{E, \_\_7\_\_\} \subseteq S_4$

The index for J is $\frac{24}{8} = 3$

There are others in K : $\alpha^2 + \gamma^2 = \sqrt{p^2 - 4q}$

$$\left.\begin{array}{ll} \text{and from} & \alpha + \beta = 0 \\ & \gamma + \delta = 0 \\ \underline{\text{also}} \text{ get} & \alpha^2 = \beta^2 \\ & \gamma^2 = \delta^2 \end{array}\right\} K_2$$

$J \longrightarrow I = \{E, \_\_3\_\_\}$ index: $I \subseteq J \quad \frac{8}{4} = 2$

There's another $\sqrt{\frac{(-p-D)}{2}}$ where

$D = \sqrt{p^2 - 4q}$.

This adds $\quad \gamma - \delta = 2\sqrt{\frac{-p-D}{2}}$ $\Big\}$ $K③$

and only $\underbrace{\begin{matrix}\alpha \to \beta \\ \beta \to \alpha\end{matrix}}$ satisfy all of $K①$ $K②$ $K③$

$$H \subseteq I \subseteq J \subseteq S_4$$

order:	2	4	8	24
index		2	2	3

Finally, $\sqrt{\frac{(-p+D)}{2}} \to \alpha - \beta = 2\sqrt{\frac{-p+D}{2}}$

$K④$

only E satisfies $K①$ $K②$ $K③$ $K④$

$$E \subseteq H$$

$$1 \quad 2$$

$$2$$

what 19 year old Evariste Galois (1811-1832)

yes! 20 years old @ death

showed was that a polynomial can be solved "in radicals" only if the indices of the composition series are prime.

n	indices
2	2
3	2, 3
4	2, 3, 2, 2
5	2, 60
6	2, 360

quintic and higher ~~we~~ are not solvable

Galois coined the term "Group".

His life? Terrible.

The next steps were seemingly unrelated..

Gauss found for the Binary Quadratic with integer coefficients

$$Ax^2 + 2Bxy + Cy^2$$

$A, B, C \in \mathbb{Z}$

"transform" ...

$$\left.\begin{array}{l} x \to \alpha x' + \epsilon y' \\ y \to \rho x' + \delta y' \end{array}\right\} T$$

to give

$$ax'^2 + 2bx'y' + cy'^2$$

where

$$\begin{aligned} a &= A\alpha^2 + 2B\alpha\rho + C\rho^2 \\ b &= A\alpha\epsilon + B(\alpha\delta + \epsilon\rho) + C\delta\rho \\ c &= A\epsilon^2 + 2B\epsilon\delta + C\delta^2 \end{aligned}$$

and

$$(b^2 - ac) = (B^2 - AC)(\alpha\delta - \epsilon\rho)^2$$

$$I(a, b, c..) = I(A, B, C...)\,\Delta^\lambda$$

called the "invariant" by James Joseph Sylvester

effectively invented "matrix" theory } in collaboration with Arthur Cayley

Lagrange found something similar for ternary quadratic forms.

Boole was the real founder — studied the general transformation of homogeneous polynomials with general coefficients.

Arthur Cayley — 1844 took it over → 960 papers.

Found a <u>general method</u> for calculation of invariants for many forms, For example

$$Ax^4 + 4Bx^3y + 6Gx^2y^2 + 4Dxy^3 + Ey^4$$

$$I = AE - 4B + 3G^2 \qquad \text{so what?}$$

Boole found another! and another!

<u>Now</u> the question was: how many were there?
how to find new ones?

→ Became the "Theory of Invariants"

BIG DEAL in late 19th century

Clebsch, Gordon, Hilbert, M. Noether, Weyl, Poincaré

Emmy Noether   thesis

Cayley's Theorem: Every finite group of order n is isomorphic to a subgroup of S_n.

Cayley formalized the Symmetric Group & Point Groups. & first established the postulates of Group Theory.

Led to Theory of Covariants

$$K(A, B, C \ldots ; x, y, z \ldots) = K'(a, b, c \ldots ; x', y', z' \ldots)\Delta$$

covariants

what kinds of transformations on forms of equations lead them to be the same...

Group Theory enters physics. through Lie and Klein for the Continuous Groups

Okay. Back to work.

DEF. Homomorphism; Two groups $\mathcal{G}$ and $\mathcal{H}$ are homomorphic if for some $h_1 \in \mathcal{H}$ can be associated with each element in $\mathcal{G}$ such that

if $g_1 g_2 = g_3 \in \mathcal{G}$

then $h_1 h_2 = h_3 \in \mathcal{H}$

D_3 and C_2 are Homomorphic

DEF. <u>Complex</u>; A complex is a set of elements from a group.

examples: D_3 complexes: $D_3' = \{d, f, e\}$
$D_3'' = \{a, b, c\}$

C_2 complexes: $C_2' = \{e\}$
$C_2'' = \{a\}$

Relations can exist among complexes

$d \circ f = e \qquad e \circ e = e \quad \Rightarrow \quad D_3' \circ D_3' = D_3'$

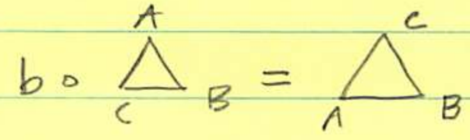
$a \circ b = d \qquad a \circ a = e \quad \Rightarrow \quad D_3'' \circ D_3'' = D_3'$

$d \circ c = b \qquad a \circ e = a \quad \Rightarrow \quad D_3' \circ D_3'' = D_3''$

DEF <u>conjugate</u>; Within a group $\mathcal{G}$, $g_i \in \mathcal{G}$ and $g_j \in \mathcal{G}$ are conjugate elements if if there exists some $g_h \in \mathcal{G}$ such that

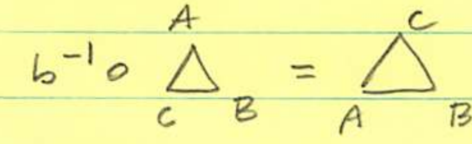
$$g_i = g_h g_i g_h^{-1}$$

example

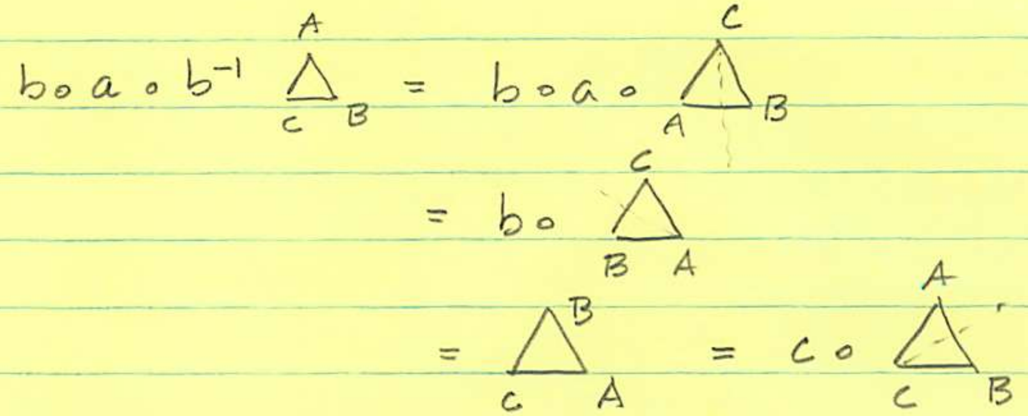


since $b \circ b^{-1} = e$ and $b \circ b = e \Rightarrow b = b^{-1}$

so



So,



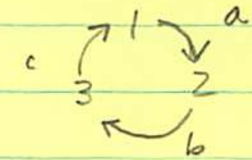
So, $c = b \circ a \circ b^{-1}$ and

c is conjugate to a

likewise d and f are conjugate

DEF. <u>class</u>; Elements which are conjugate to one another are together elements of a class. Each elements belongs to one class.

How about S_3.



, hence, "cyclic"

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$p_a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad p_b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_c = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$p_d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

more conventional notation..

$$p_d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} = (1\,3\,2)$$

move them so that the cycles close as adjacents

So, an element of S_8:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 & 5 & 7 & 8 \\ 2 & 3 & 1 & 6 & 4 & 7 & 5 & 8 \end{pmatrix}$$

$$= (123)(46)(57)(8)$$

The S_3 / D_3 elements can be written

D_3	e	a	b	c	d	f
S_3	e	(12)	(23)	(13)	(132)	(123)

This separates out the classes $\mathcal{C}_1$: e

$\mathcal{C}_2$: d, f or (132), (123)

$\mathcal{C}_3$: a, b, c or (12), (23), (13) Before going onto Group Representations .. some notation and summary.

Notational confusion - alert!

Let's distinguish, between a set of OPERATIONS and the combination of operations that create particular groups

Symmetry operation → transformation of some object that leaves it unchanged.

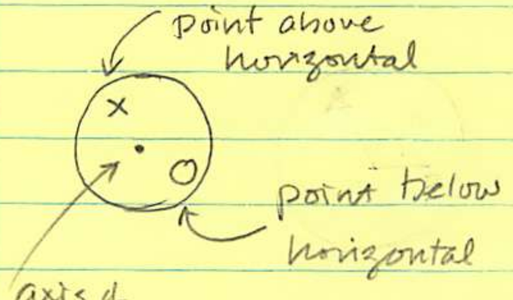
They conventionally are:

$C_n, \sigma_h, \sigma_v, \sigma_d, S_n, i$

operation C_n: (n - integer) a rotation of $\theta = \frac{2\pi}{n}$ about some axis in space

n: <u>multiplicity</u> of the axis

considered to be vertical → <u>principle</u> <u>axis</u>; axis with highest n for body with >1 axis of symmetry



axis of symmetry. If principle — vertical

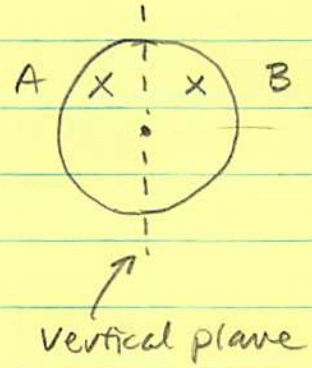
axis

Operation C_n

N.B. $C_n^k \to \frac{2\pi k}{n}$ you'll see...

operations σ_d and σ_v: Consider a vertical plane <u>including</u> the principle axis, the σ_v operation is a reflection across that plane

σ_v:

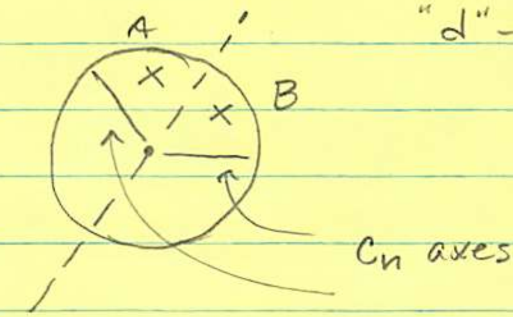


operation

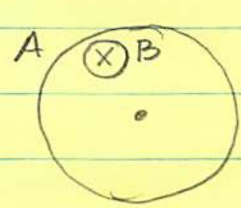
σ_v

σ_d: is in conjunction with C_n

"d" – "diagonal"



operation σ_h: reflection through horizontal plane $\perp$ principle axis



operation S_n : an <u>improper rotation</u>

a rotation about a symmetry axis followed by reflection σ_h or visa versa.

yes. there is a notational confusion – <u>not</u> the symmetric Group

$$S_n = C_n \sigma_h = \sigma_h C_n$$

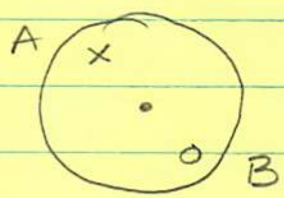
n must be even

$\theta = \frac{2\pi}{n}$

operation S_n

operation i : <u>inversion</u>, takes Cartesian coordinates $(x, y, z) \to (-x, -y, -z)$

$\equiv S_2$



These always leave at least one point unmoved.

Now, the actual POINT GROUPS → combinations of the above operations.

Notation: "Schoenflies" ✓

also "international"

also ... some others

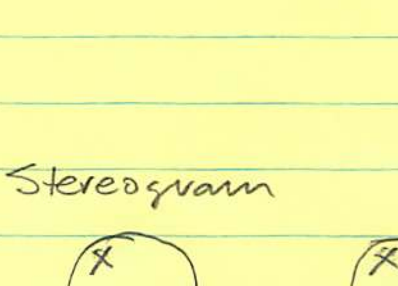
GROUPS

$\boxed{C_n}$ Successive application of operations C_n.

Elements: $C_n, C_n^2, C_n^3 \dots e \equiv C_n^n$

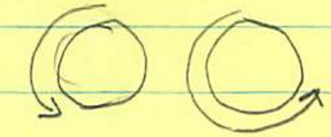
so, we had C_2 ... $\{ C_2, C_2^2 \equiv e \}$

$\frac{2\pi}{2} = \pi$ $\quad \frac{2\pi \cdot 2}{2} = 2\pi$

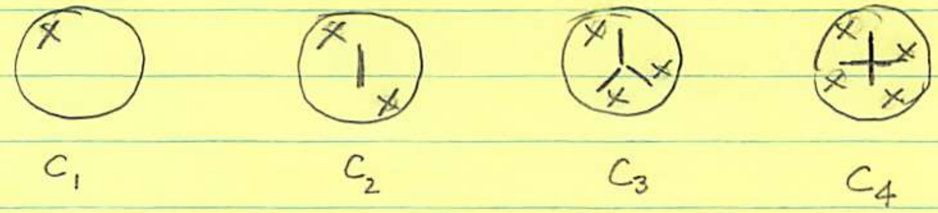


C_3 ... $\{ C_3, C_3^2, C_3^3 \equiv e \}$

$\frac{2\pi}{3} = 120^\circ$ $\quad \frac{2\pi \cdot 2}{3} = 240^\circ$ $\quad \frac{2\pi \cdot 3}{3} = 2\pi$

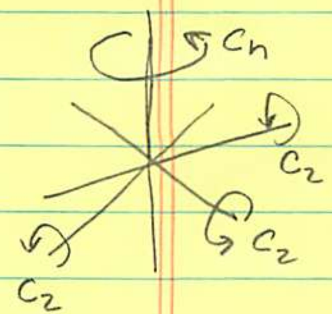


Stereogram



C_1 $\quad C_2$ $\quad C_3$ $\quad C_4$

$\boxed{D_n}$ Dihedral Group order n



1 n-fold axis — principle — C_n

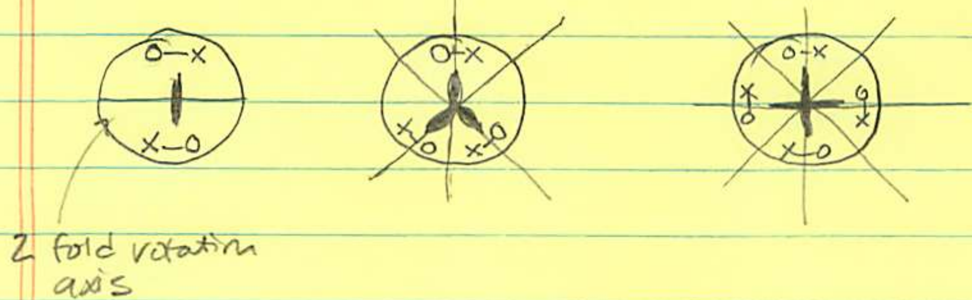
n 2-fold axes — symmetrically arranged
n, C_2's

$2n$ total elements

the operations "covering" a regular polygon.

we did D_3 $\{a, b, c, d, f, e\}$

$\{C_2^{(1)}, C_2^{(2)}, C_2^{(3)}, C_3, C_3^2, C_3^3\}$

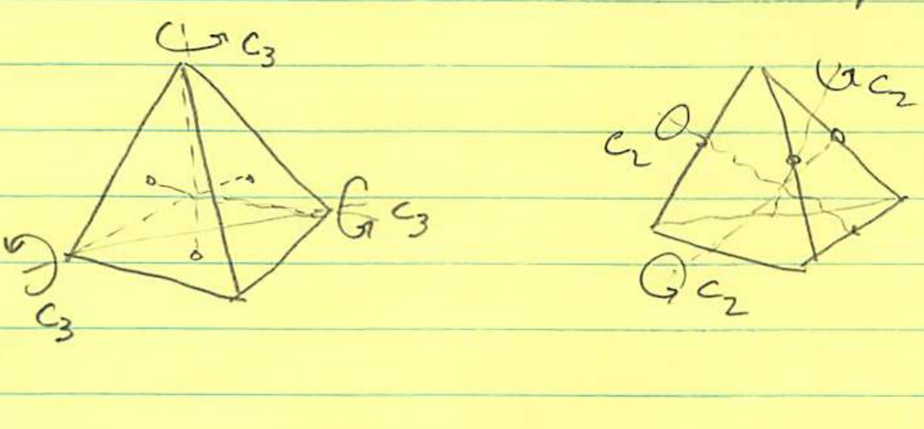


D_2 D_3 D_4

$\boxed{T}$ symmetry operations on regular tetrahedron

4 3-fold axes — through each apex & face center

3 2-fold axes — through centers, opposite sides

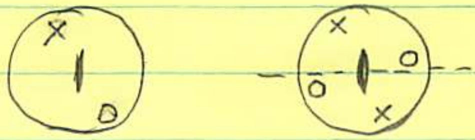


$\boxed{O}$ Octahedral group. (another notational problem with the Orthogonal group)

$\boxed{S_n}$ or sometimes $\boxed{S_{2n}}$ NOT the Symmetric Group! (which is sometimes called the Permutation Group $\mathcal{S}_n$ or P_n)

Successive application of $\{S_n, S_n^2 \cdots S_n^n = e\}$

n-fold improper rotations



S_2 S_4

S_n is actually a combination of a C_2 rotation and $\sigma_{h,v}$. (problem)

$\boxed{C_{nh}}$ n-fold rotation axis + reflection plane $\perp$ to the axis.

It is a larger group than C_n — each element is multiplied by an element of a reflection

DEF: direct product group; If $\mathcal{K}$ is a group containing subgroups $\mathcal{H}$ and $\mathcal{G}$, & each $k \in \mathcal{K}$ is a product $g \circ h$, then $\mathcal{K}$ is a direct product group.
$\mathcal{K} = \mathcal{H} \otimes \mathcal{G}$

$C_{nh} = C_n \otimes C_{1h}$

C_{1h} C_{2h} C_{3h}

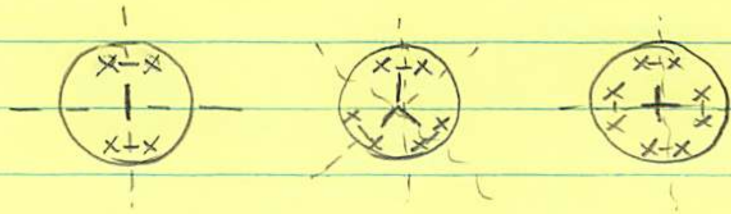
where C_{1h} is a group with 2 elements $\{e, \sigma_h\}$

$\boxed{C_{nv}}$

Symmetry operations with 1 n-fold principle axis and n reflection planes including the principle axis - 2n elements

$C_{nv} = C_n \otimes C_{1v}$

etc.



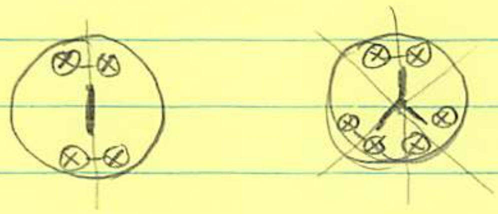
C_{2v} C_{3v} C_{4v}

$\boxed{D_{nh}}$

Symmetry of a regular prism

$D_{nh} = D_n \otimes C_{1h}$ 2n elements

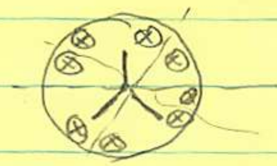
if n-even $D_{nh} = D_n \otimes S_2$



etc

D_{2h} D_{3h}

$\boxed{D_{nd}}$



etc

D_{2d}