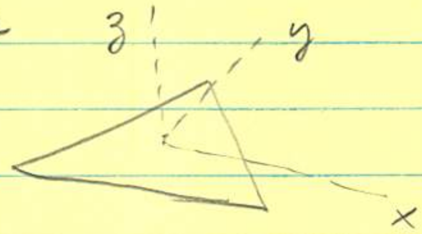
Vector spaces

Groups have applicability in physics when they have an algebraic "playing field" to operate in.

For example -- the triangle needs to be embedded in a coordinate space



with the operations defined as transformations on x, y, z.

GROUP REPRESENTATIONS

→ vector spaces. $\vec{x} = (x, y, z)$ is an example. But, it's more general than that.

→ like the notation from Tung's book.

Again, a 19^{th} century invention -- largely from Grassmann, a high school teacher

So, we have 2 entities

I. $\boxed{\text{Scalars}}$ $a, b, c \ldots$ field, $\mathcal{F}$ $\left\{ \begin{array}{ll} A, & \oplus \\ B, & \otimes \end{array} \right.$ + properties

II. $\boxed{\text{vectors}}$ a vector space, $\mathcal{V}$

with a set of elements $\xi, \eta, \ldots$ having properties:

A. ξ & $\eta \in \mathcal{V} \Rightarrow \exists$ another $\underbrace{\xi \oplus \eta}_{\text{vector sum}} \in \mathcal{V}$ closure

& commutative (B), associative (C)

origin (D), inverse (E)

$\exists \phi \ni$
$\xi \oplus \phi = \xi$

F. for $a \in \mathcal{F}$ and $\xi \in \mathcal{V}$ $\exists$ $a\xi \in \mathcal{V}$

& associative (G), $\mathbb{1}\xi = \xi$ (H), distributive with respect to vector addition and also scalar addition (I) J

DEF. span: A space $\mathcal{V}$ is spanned by a set of vectors if every vector $\in \mathcal{V}$ can be represented by a linear combination

A - J above define a Linear Vector Space, L

III. $\mathcal{V}$ is Unitary if $\exists$ an operation called the scalar product, or "inner product"

$(\xi, \eta) \rightarrow$ scalar-valued, real or complex

A. $(\xi, \eta) = (\eta, \xi)^* = a$

B. $(\xi, \eta \oplus \gamma) = (\xi, \eta) \oplus (\xi, \gamma)$

C. $(\xi, a\eta) = a(\xi, \eta)$

D. 1) $(\xi, \xi) \geq 0$ 2) $(\xi, \xi) = 0$ iff $\xi = \phi$

E. 1) $(\xi, a\eta \oplus b\gamma) = a(\xi, \eta) \oplus b(\xi, \gamma)$ linear

2) $(a\xi \oplus b\eta, \gamma) = a^*(\xi, \gamma) \oplus b^*(\eta, \gamma)$ antilinear

IV also: "outer products"

NOTATION: we'll use Dirac Notation

$$\xi \leftrightarrow |\xi\rangle$$

for both linear and antilinear spaces

$\mathcal{V}$ $\qquad\qquad$ $\tilde{\mathcal{V}}$

$$(\xi, \eta)$$

antilinear in ξ $\qquad$ linear in η

So: $\left.\begin{array}{l} |\eta\rangle \in \mathcal{V} \\ \langle\xi| \in \tilde{\mathcal{V}} \end{array}\right\}$ $\tilde{\mathcal{V}}$ is "dual" to $\mathcal{V}$

If ξ and η are antilinear to one another, then

$$(\xi, \eta) = \langle \xi | \eta \rangle = \langle \eta | \xi \rangle^{*}$$

Useful when a Representation is chosen ... then, the vectors are Representatives.

IV. Usually.. think of an ordered list

$$\xi = \{ \underbrace{\xi^1, \xi^2, \xi^3, \ldots \xi^n}_{\text{components: ordered list of scalars}}) \quad \text{a contravariant vec.}$$

so, $\quad \xi \oplus \eta = \gamma$

i $\quad \{ \xi^1 \oplus \eta^1, \xi^2 \oplus \eta^2, \ldots \} = \{ \gamma^1, \gamma^2 \ldots \}$

ii $\quad a\xi = \{ a\xi^1, a\xi^2, \ldots \}$

iii $\quad \langle \xi | \xi \rangle \equiv |\xi|^2 = |\xi^1|^2 + |\xi^2|^2 + \ldots |\xi^n|^2$

where $\quad |\xi^i|^2 \equiv \xi^{i\times} \xi^i$

think "geometry": $\quad |\xi| = +\sqrt{\langle \xi | \xi \rangle}$ "length"

if $\quad |\xi| = \mathbb{1} \Rightarrow \xi$ is "normalized"

V. Scalar product

$$\langle \xi | \eta \rangle = \sum_{i=1}^{n} \xi_i^{+} \eta^i \quad (= (\xi, \eta))$$

where $\quad \xi_i^{+} = \xi^{i\times}$

VI Einstein summation convention:

$$\sum_{i=1}^{n} \xi_i^{+} \eta^i \equiv \xi_i^{+} \eta^i$$

VII Orthogonality $\quad \langle \xi | \eta \rangle = 0 \Rightarrow \xi$ & η are orthogonal more notation:

We can have sets of vectors

$\vec{\phi}_i$ $\qquad\qquad$ $\vec{\phi}_1 = \vec{\alpha}$

index labels different vectors $\qquad$ $\vec{\phi}_2 = \vec{\beta}$

$\vdots$

each a vector with components

So, we <u>could</u> have $\quad \phi_{ij}$

vector $\quad$ component

or

$$|\phi_{(i)}\rangle^{j}$$

component.

vector .. will use "( )" if not obvious

VIII. Linear independence of a set of vectors

$$|\phi_{(i)}\rangle a^i = 0 \quad \Rightarrow \quad a^i = 0 \text{ for all } i.$$

Without proof:

a. for d-dimensional V, no more than d linearly independent vectors spanning V

b. It is possible to form a mutually orthogonal set of vectors spanning V which are L.I.

c. no such set is unique.

eg.

The most familiar vector set : $\mathcal{E}_3$

$\hat{i} = \{1, 0, 0\}$ $\quad \hat{j} = \{0, 1, 0\}$ $\quad \hat{k} = \{0, 0, 1\}$

They are mutually orthogonal $\quad \langle i | j \rangle = 0$ etc

In my notation $\quad |\hat{e}_{(i)}\rangle$: $\quad |\hat{e}_{(1)}\rangle = \hat{i} = \{1, 0, 0\}$

$$|\hat{e}_{(2)}\rangle = \hat{j}$$

$$|\hat{e}_{(3)}\rangle = \hat{k}$$

so,

$$|\hat{e}_{(1)}\rangle = \{e^1_{(1)}, e^2_{(1)}, e^3_{(1)}\} = \{1, 0, 0\}$$

get it?

Orthonormality – conveniently, one statement

$$\langle \hat{e}^{(j)} | \hat{e}_{(i)} \rangle = \delta^j{}_i \quad = 0 \quad i \neq j$$

$$= 1 \quad i = j$$

Let $|\eta_1\rangle, |\eta_2\rangle, \ldots |\eta_d\rangle$ be a set of L. I. vectors in V_d and $|\xi\rangle$ be an arbitrary vector in V_d

$|\xi\rangle$ has a unique representation as a linear combination of $|\eta_i\rangle$

$$|\xi\rangle = |\eta_{(i)}\rangle c^i$$

the "basis set"

We can determine the c^i :

$$\langle \eta^{(j)} | \xi \rangle = \langle \eta^{(j)} | \eta_{(i)} \rangle c^i$$

$$= \delta^j{}_i \, c^i = \boxed{c^j = \langle \eta^{(j)} | \xi \rangle}$$

So, $$|\xi\rangle = |\eta_{(i)}\rangle \underbrace{\langle \eta^{(i)} | \xi \rangle}$$

geometrical notion of projecting $\vec{\xi}$ along $\vec{\eta}$'s

Within the particular η basis, the components of ξ are:

$$\langle \eta^{(i)} | \xi \rangle \equiv \xi^i = c^i$$

$$|\xi\rangle_\eta = \{\xi^1, \xi^2, \ldots \xi^d\} = \{ \langle \eta^{(1)} | \xi \rangle, \langle \eta^{(2)} | \xi \rangle \ldots \}$$

So, we would write

$$|\xi\rangle = |\eta_{(i)}\rangle \xi^i$$

If we had a $\boxed{\text{different basis}}$ – a change of coordinate system – $|\xi\rangle$ can be represented by it as well

$$|\xi\rangle = |e_k\rangle d^k = |e_k\rangle \xi^{k'}$$

To calculate the components again, project

$$\langle e^m | \xi \rangle = \langle e^m | e_n \rangle \xi^{n} = \delta^m_{\ k}\, \xi^{k'} = \xi^{m'} \qquad ⓪$$

As before $\quad |\xi\rangle = |e_n\rangle\langle e^n|\xi\rangle \quad = |e_n\rangle \xi^{n'}$

Same vector, different representations

$$|\xi\rangle = |\eta_i\rangle\langle\eta^i|\xi\rangle \qquad ① \quad = |e_n\rangle \xi^{n'}$$
$$= |\eta_i\rangle\langle\eta^i|e_\ell\rangle \xi^{\ell'} \qquad ②$$

Expand one basis in the other

$$|\eta_{(i)}\rangle = |e_{(n)}\rangle a^n_{\ (i)} \qquad \text{or}$$

$$= |e_{(n)}\rangle n^n_{\ (i)} \qquad ③$$

or

$$\langle \eta^{(i)}| = \langle e^{(n)}| n^{(i)}_{\ \ n} \qquad ④$$

⓪ ①

From ① $\quad |e_{(k)}\rangle \xi^{k'} = |\eta_{(i)}\rangle\langle e^{(n)}| n^{(i)}_{\ \ n} |e_{(\ell)}\rangle \xi^{\ell'}$

④

$|e_{(n)}\rangle \xi^{n'} = |\eta_i\rangle\langle\eta^i|\xi\rangle$

$= |\eta_i\rangle\langle e^{(n)}| n^{(i)}_{\ \ n} |e_\ell\rangle \xi^{\ell'}$

$$= |\eta_{(i)}\rangle\langle e^{(n)}|e_{(\ell)}\rangle n^{(i)}_{\ \ n}\, \xi^{\ell'}$$
$$= |\eta_{(i)}\rangle \delta^n_{\ \ell}\, \xi^{\ell'} n^{(i)}_{\ \ n}$$

$$= |\eta_{(i)}\rangle n^{(i)}_{\ \ \ell}\, \xi^{\ell'}$$

But

$$|e_{(n)}\rangle \xi^{n'} = |e_{(n)}\rangle \underbrace{n^{n}{}_{(i)}\, n^{(i)}{}_{\ell}}_{\text{must be } \delta^{n}{}_{\ell}} \xi^{\ell'}$$

③

$$|e_{(n)}\rangle \xi^{n'} = |e_{(n)}\rangle \xi^{n'}$$ ← Σ in effect, so indices are dummy

$$\sum_i n^{n}{}_{(i)}\, n^{(i)}{}_{\ell}$$ components,

vector

$$= \frac{\delta^{n}{}_{\ell}}{\langle \eta^{(i)} | \eta_{(i)} \rangle}$$ ← normalized for generality

This is the condition for Completeness or Closure of the basis set $|\eta_{(i)}\rangle$

Two Basis Sets: $|\eta_i\rangle$ and $|\gamma_j\rangle$

$$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle \eta^{j}{}_{(i)} \qquad |\gamma_{(j)}\rangle = |\eta_{(k)}\rangle \gamma^{k}{}_{(j)} \qquad ①$$

$$= |\gamma_{(j)}\rangle N^{j}{}_{i} \qquad\qquad = |\eta_{(k)}\rangle G^{k}{}_{j}$$

$$|\eta_{(i)}\rangle = |\eta_{(k)}\rangle G^{k}{}_{j} N^{j}{}_{i}$$

Put in the Σ

$$|\eta_{(i)}\rangle = \sum_k \sum_j |\eta_{(k)}\rangle G^k{}_j N^j{}_i$$

$$= \sum_k |\eta_{(k)}\rangle \underbrace{\sum_j G^k{}_j N^j{}_i}$$

$$\sum_j G^k{}_j N^j{}_i = \delta^k{}_i \longleftarrow \text{need this to be } \delta^k{}_i$$

notice this looks like matrix multiplication

$$\mathbb{G}\mathbb{N} = \mathbb{I} \quad \underline{\text{or}} \quad \mathbb{G} = \mathbb{N}^{-1} \qquad \mathbb{N} = \mathbb{G}^{-1}$$

notation! $M^i{}_j$

i ↑ row

j ↑ column

and the transpose $\quad M^T{}_i{}^j = M^j{}_i$

Had this been the duals -

$$\langle \eta^{(i)}| = \sum_j \underbrace{(G^j{}_i)^*}_{\text{indices aren't right!}} \langle \gamma^{(j)}|$$

DEFINE! $\quad (M^j{}_i)^* \equiv M^*_j{}^i \equiv M^{+i}{}_j \qquad$ Tung convention Appendix A

So, $\langle \eta^{(i)}| = \sum_j G^{*}_{j}{}^{i} \langle \gamma^{(j)}|$

Form $\langle \gamma^{(k)} | \eta_{(i)} \rangle = \sum_j \underbrace{\langle \gamma^{(k)} | \gamma_{(j)} \rangle}_{\text{O.N.}} N^j{}_i$ ①

component of $\gamma^{(k)}$
along $\eta_{(i)}$ $= \sum_j \delta^k{}_j N^j{}_i = N^k{}_i$

Notice $\langle \eta^{(k)} | \eta_{(i)} \rangle = \delta^k{}_i = \sum_j \langle \eta^{(k)} | \gamma_{(j)} \rangle N^j{}_i$

$= \sum_j \langle \gamma^{(j)} | \eta_{(k)} \rangle^* N^j{}_i$

$= (N^j{}_k)^* N^j{}_i$

$= N^{+k}{}_j N^j{}_i = \delta^k{}_i$

SO: $\mathbb{N}^+ \mathbb{N} = \mathbb{I}$

When the bases are orthonormal, $\mathbb{N}$ and $\mathbb{G}$ are unitary.

$\mathbb{N}^+ = \mathbb{N}^{-1}$

$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle N^j{}_i$

$|\gamma_{(i)}\rangle = |\eta_{(i)}\rangle N^{+i}{}_j$

Expand an arbitrary vector

substitute

$|\xi\rangle = |\eta_{(i)}\rangle \xi^i$

$= \underline{\underline{|\gamma_{(j)}\rangle \xi^{j'}}}$

$|\xi\rangle = |\gamma_{(j)}\rangle N^j{}_i \xi^i$

Compare $=$ with $\equiv$

$$|\xi\rangle = |\gamma_{(j)}\rangle \xi^{j'} = |\gamma_{(j)}\rangle N^j{}_i \xi^i \Rightarrow N^j{}_i \xi^i = \xi^{j'}$$
$$N^{+i}{}_j \xi^{j'} = \xi^i$$

$\rightarrow$ the same operation that connects the bases, also connects components of arbitrary vectors

$\boxed{\text{EXAMPLE}}$ $\mathcal{E}_3$

$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$$

now, in the new language

$$|x\rangle = |\hat{e}_{(j)}\rangle x^i$$

or $$|e_{(i)}\rangle = |e_{(j)}\rangle e^j{}_{(i)}$$

				$i=$
$\hat{i}$	$e^1_{(1)} = 1$	$e^2_{(1)} = 0$	$e^3_{(1)} = 0$	1
$\hat{j}$	$e^1_{(2)} = 0$	$e^2_{(2)} = 1$	$e^3_{(2)} = 0$	2
$\hat{k}$	$e^1_{(3)} = 0$	$e^2_{(3)} = 0$	$e^3_{(3)} = 1$	3

which can be represented as

$$e_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$e^{(1)} = (1, 0, 0) \qquad \text{etc.}$$

Project an arbitrary vector

$$\langle e^{(j)} | x \rangle = e^{(j)}{}_{\ell} \langle e^{(\ell)} | x \rangle$$
$$= e^{(j)}{}_{\ell} \langle e^{(\ell)} | e_{(i)} \rangle x^{i}$$

$$= \delta^{\ell}{}_{i}\, e^{(j)}{}_{\ell}\, x^{i} = \underbrace{e^{(j)}{}_{i}\, x^{i}}_{\text{terms for dot product}}$$

So, for example

$$\langle e^{(1)} | x \rangle = \hat{i} \cdot \vec{x}$$
$$= e^{(1)}{}_{1} x^{1} + e^{(1)}{}_{2} x^{2} + e^{(1)}{}_{3} x^{3}$$
$$= 1 \cdot x^{1} + 0 \cdot x^{2} + 0 \cdot x^{3} = x^{1}$$

In this same way

$$x^{i} = \begin{pmatrix} x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

$$\langle e^{(j)} | x \rangle = e^{(j)}{}_{i}\, x^{i}$$

$$= (1, 0, 0) \begin{pmatrix} x^{1} \\ x^{2} \\ x^{3} \end{pmatrix} = x^{1}$$

This easily extends to arbitrary vectors $\vec{y} \cdot \vec{x}$

$$\langle y | x \rangle = y_j \langle e^{(j)} | e_{(i)} \rangle x^i$$

$$= \delta^j{}_i \, y_j x^i = y_i x^i \Rightarrow$$

$$(y_1 \, y_2 \, y_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

SUBSPACES

Consider an n-dimensional vector space, V_n.
It consists of all linear combinations of n-linearly independent vectors — each spans V_n.

Choose a subset $m < n$ .. and another set of L.I. vectors spans $V_m \in V_n$

The set which is orthogonal to those in V_m creates another $V_m^\perp$ which is (n-m dimensional)

V_n is decomposed $\quad V_n = V_m \oplus V_m^\perp$

VIII Generally, two arbitrary spaces can be combined into a direct sum:

$$V_{n+m} = \underbrace{V_n^a \oplus V_m^b}_{\text{subspaces}}$$

which has properties

A. vectors combine as:

$$|\xi\rangle = |\eta(a)\rangle \oplus |\eta(b)\rangle$$

B. elements of $|\xi\rangle$ $\quad (|\eta(a)\rangle ; |\eta(b)\rangle)$
another set from the
subspaces can combine

$$(|\eta(a)\rangle ; |\eta(b)\rangle) \oplus (|\gamma(a)\rangle ; |\gamma(b)\rangle)$$
$$= (|\eta(a)\rangle \oplus |\gamma(a)\rangle ; |\eta(b)\rangle \oplus |\gamma(b)\rangle)$$

notice $\quad (|\eta(a)\rangle ; |\phi\rangle) \rightarrow$ isomorphic to $\mathcal{V}_n^a$

origin

We can also form a direct product space

$$\mathcal{V}_{n,m} = \mathcal{V}_n^a \otimes \mathcal{V}_m^b$$

with properties:

A. $|\xi\rangle \in \mathcal{V}_{n,m} = |\eta(a)\rangle \otimes |\eta(b)\rangle$

B. $(c\,|\eta(a)\rangle \otimes |\eta(b)\rangle) = c\,(|\eta(a)\rangle \otimes |\eta(b)\rangle)$

C. $\eta(a) \otimes (\xi(b) \oplus \gamma(b)) = \eta(a) \otimes \xi(b) \oplus \eta(a) \otimes \gamma(b)$

D. Suppose $|\alpha_{(i)}(a)\rangle$ is basis in V^a

$|\beta_{(j)}(b)\rangle$ is " V^b

then: $\underset{V_n^a}{|\alpha_{(i)}(a)\rangle} \otimes \underset{V_m^b}{|\beta_{(j)}(b)\rangle}$ forms an $n \times m$ dimensional basis set in $V_{n,m}$

Expansion: $|\xi(a)\rangle = |\alpha_{(i)}(a)\rangle a^i = |\alpha_{(i)}(a)\rangle \xi^i_{[\alpha];a}$

$|\rho(b)\rangle = |\beta_{(j)}(b)\rangle b^j = |\beta_{(j)}(b)\rangle \rho^j_{[\beta];b}$

So product

$$|\xi(a)\rangle \otimes |\rho(b)\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} |\alpha_{(i)}(a)\rangle \otimes |\beta_{(j)}(b)\rangle \xi^i_{[\alpha];a} \rho^j_{[\beta];b}$$

$\Big\downarrow$ notation

$|\xi(a)\rho(b)\rangle$

For an inner product, the spaces stay separate

$$\langle \gamma(a)\varepsilon(b) | \xi(a)\rho(b)\rangle = \langle \gamma(a)|\xi(a)\rangle \langle \varepsilon(b)|\rho(b)\rangle$$

∞ sized vector spaces are important – countably ∞ or continuously ∞.

X. A <u>compact</u> <u>space</u> is defined by

Suppose for any vector $\xi \in V$ there is a series $|\eta_{(i)}\rangle$ with the property that $\exists$ at least one $\eta_{(i)}$ with

$$|\xi - \eta_{(i)}| < \varepsilon \qquad \varepsilon \text{ arbitrarily small.}$$

Most prominent among such spaces:

set of square-integrable functions of a real variable $\mathbb{L}^2(a,b)$

Suppose we have 2 functions $f(x)$ and $g(x)$ defined on $a \le x \le b$ for the continuous variable x.

- can combine them $h(x) = f(x) + g(x)$ and get another function

- can form a scalar product

$$(f\,g) \equiv \int_a^b f^*(x)\, g(x)\, dx$$

with norm $N_f = (f\,f) = \int_a^b f^*(x) f(x)\, dx$

$$= \int_a^b |f(x)|^2 dx$$

if the N_f is finite $\Rightarrow$ "square integrable"

if $(f, g) = 0 \Rightarrow$ "orthogonal functions"

We can construct functional bases $\in \mathbb{L}^2_1(a,b)$ which can be <u>countably</u> <u>infinite</u>

and could be orthonormal:

$$(\eta^{(i)}, \eta_{(j)}) = \int_a^b \eta^{i*}(x)\, \eta_j(x)\, dx$$

$$= \delta^i{}_j$$

Can represent an arbitrary function

$$\xi(x) = \sum_{i=1}^{\infty} \eta_{(i)}(x)\, \xi^i_{[\eta]}(x) \qquad ①$$

with coefficients which are found by

$$\Delta = \int_a^b \Big| \xi(x) - \sum_{i=1}^{n} \eta_{(i)}(x)\, \xi^i_{[\eta]} \Big|^2 dx \qquad \text{drop } [\eta]$$

$$= \int_a^b |\xi(x)|^2 dx - \int_a^b \sum_i \xi^*(x)\, \eta_{(i)}(x)\, \xi^i\, dx$$

$$- \int_c^b \sum_i \eta^{(i)*}(x)\, \xi_i^*(x)\, dx$$

$$+ \int \sum\sum \eta^*_{(j)}(x)\, \eta_{(i)}(x)\, \xi^{*j} \xi_i\, dx$$

$$= N_\xi - \sum_i \left[(\xi, \eta_{(i)}) \xi^{i} + (\eta^{(i)}, \xi) \xi^*_i - \xi^*_i \xi^i \right]$$

minimize Δ $\qquad \dfrac{d\Delta}{d\xi^*} = 0 = (\eta^{(i)}, \xi) - \xi^i$

so $\qquad \xi^i = (\eta^{(i)}, \xi)$

and coefficients are seen to be the "best fit" by this condition:

$$\xi^i = \int_a^b \eta^{*(i)}(x)\, \xi(x)\, dx \qquad ①$$

XI. Completeness essentially says

$$\Delta \to 0 \quad \text{as} \quad n \to \infty$$

$$\lim_{n\to\infty} \int_a^b \left| \xi - \sum_{i=1}^n \eta_{(i)} \xi^i \right|^2 dx = 0$$

A vector space with II, III, V, X, XI are the definition of a Hilbert Space.

notice $\qquad \xi(x) = \sum_{i=1}^n \eta_{(i)}(x)\, \xi^i_{[\eta]}$ $\qquad ①$

$$= \sum_{i=1}^n \eta_{(i)}(x) \int_a^b \eta^{*(i)}(x')\, \xi(x')\, dx'$$

$$= \int_a^b \left\{ \sum_i \eta_{(i)}(x)\, \eta^{*(i)}(x') \right\} \xi(x')\, dx'$$

which provides a definition of the Dirac δ function

$$\sum_{i=1}^{\infty} \eta_{(i)}(x)\,\eta^{*j}(x') \equiv \delta(x-x')$$

and is the Closure or Completeness relation

This whole field came about from a series of lectures by Hilbert in 1925-1927, Von Neumann was his post doc, so the direct connection to formalizing quantum mechanics is clear. Richard Courant was Hilbert's student and wrote Courant and Hilbert based on those notes.