Lecture 4

What we've done so far:

(1) Vector spaces — Finite

$$|\xi\rangle = |\eta_{(i)}\rangle \, \xi^{i}_{[\eta]}$$

components of $|\xi\rangle$ in $|\eta\rangle$

abstract object of V

basis set which span V

notation $\quad |\eta_{(i)}\rangle \quad$ set of vectors (i)

$\quad n^{j}_{(i)} \quad$ the j^{th} component of the i^{th} vector

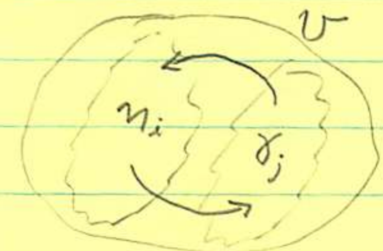
completeness:

$$\sum_i n^{l}_{(i)} \, n^{(i)}_{\ \ m} = \delta^{l}_{\ m}$$

sum over basis states — all of them.

→ you've all used completeness to "insert a complete set of states" in Quantum Mechanics

I talked about transforming from one basis set to another



$$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle N^j{}_i$$

$$|\gamma_{(j)}\rangle = |\eta_{(k)}\rangle G^k{}_j$$

Back and forth $\Rightarrow$ $\displaystyle\sum_j G^k{}_j N^j{}_i = \delta^k{}_i$

$\Downarrow$ matrix multiplication

$$\mathbb{G}\,\mathbb{N} = \mathbb{1}$$

I used Wu-Ki's notation (Appendix I! not "A"!)

$$M^i{}_j$$

↑ row ↑ column

where the transpose $M^j{}_i \xrightarrow{\text{transpose}} M^T{}_i{}^j$

and from the dual transformation,

$$\langle \eta^{(i)}| = \sum_j \underbrace{(G^j{}_i)^*}_{\text{nope}} \langle \gamma^{(j)}|$$

$$(M^j{}_i)^* \equiv M^*_j{}^i \equiv M^{\dagger i}{}_j$$

I defined direct sum spaces
direct product spaces
↑
yes, mathematicians use "tensor product"

→ subspaces.

② Vector spaces – functions of continuous variable
function spaces.

$$\xi(x) = \sum_{i=1}^{\infty} \eta_{(i)}(x)\, \xi^{i}_{[\eta]}(x)$$

↑ countably ∞ # basis functions

↖ components

$$\xi^{i} = \left(\eta^{(i)}, \xi\right) = \int_a^b \eta^{*(i)}(x)\, \xi(x)\, dx$$

completeness:

$$\sum_{i=1}^{\infty} \eta_{(i)}(x)\, \eta^{*(i)}(x') = \delta(x-x')$$

→ Hilbert Spaces

Many basis sets: H_n, P_n, C_n, $L_{n\ell}$, etc.

Square-Integrable, like $\mathbb{L}^2(a,b)$.

(3) Vector spaces - Continuous

Some are not — they are defined over $[-\infty, +\infty]$.
Examples are the integral transforms, like Fourier Trans.

$$g(p) = \sqrt{1/2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ipx/\hbar}\, dx$$

$$f(x) = \sqrt{1/2\pi} \int_{-\infty}^{\infty} g(p)\, e^{ipx/\hbar}\, dp$$

A basis can be defined

$$\eta_p(x) = \sqrt{1/2\pi}\, e^{-ipx/\hbar}$$

continuous "index" now

so that

$$g(p) = \int_{-\infty}^{\infty} \eta_p(x)\, f(x)\, dx$$

$$f(x) = \int_{-\infty}^{\infty} \eta_p^*(x)\, g(p)\, dp$$

Completeness here requires

$$\int dp\, \eta_p(x)\, \eta_p^*(x) = \int \frac{dp}{2\pi}\, e^{-ip(x-x')/\hbar}$$

$$= \delta(x-x')$$

So, notation for functional bases -- discrete and continuous -

Discrete Basis: $|\xi(x)\rangle = \underbrace{\sum_{i=1}^{\infty}}_{\text{countable set}} |\eta_{(i)}(x)\rangle \xi^{i}_{[\eta]}$

Continuous Basis: $|\xi(x)\rangle = \underbrace{\int_{\Omega_x}}_{\text{continuous set}} d\omega \, |\gamma_x(\omega)\rangle \xi_{[\gamma]}(\omega)$

For each

Completeness | Orthogonality

D. $\sum_{i=1}^{\infty} |\eta_{(i)}(x)\rangle\langle \eta^{(j)}(x')| = \delta(x-x')$ $\quad$ $\langle \eta^{(j)} | \eta_{(i)}\rangle = \delta^{j}{}_{i}$

C. $\int_{\Omega_x} dx \, |\gamma_x(\omega)\rangle\langle \gamma_x(\omega')| = \delta(\omega-\omega')$ $\quad$ $\langle \gamma_x(\omega) | \gamma_x(\omega')\rangle = \delta(\omega-\omega')$

can always construct bases which will be orthogonal

Remember.. we have two spaces going on here!

a function space (in which wave functions live in QM)

a completely abstract Hilbert space of state vectors.

$|\xi\rangle$ ← can represent it in various ways by projecting

Let's say for some Hilbert space vector $|p\rangle$ we assign a basis function

$$\eta_p(x) = \sqrt{1/2\pi}\, e^{ipx}$$

Just like $\quad |\xi(x)\rangle = \sum_{i=1}^{\infty} |\eta_{(i)}(x)\rangle\, \xi^i_{[\eta]}$

we can expand in this continuous basis

$$|\xi\rangle = \int dp'\, |p'\rangle\, \tilde{\xi}(p') \qquad \text{spectral expansion}$$

By

$$\langle p|\xi\rangle = \int dp'\, \langle p|p'\rangle\, \tilde{\xi}(p')$$

$$\langle p|p'\rangle = \int dx\, \eta^*_{p'}(x)\, \eta_p(x)$$

$$= \frac{1}{2\pi}\int dx\, e^{-ix(p'-p)} = \delta(p'-p)$$

so $\langle p | \xi \rangle = \int dp' \, \delta(p'-p) \, \tilde{\xi}(p')$

$$\langle p | \xi \rangle = \tilde{\xi}(p)$$

project arbitrary state vector on to Hilbert space $\langle p|$

get a function of p

Do it again, but now define another basis function

$$\xi_{x'}(x) \equiv \delta(x - x')$$

continuous index

not square integrable!

Like before

$$|\xi\rangle = \int dx' \, |x'\rangle \, \xi(x')$$

$$\langle x | \xi \rangle = \int dx' \, \langle x | x' \rangle \, \xi(x')$$

where $\langle x | x' \rangle = \int dx'' \, \xi^*_x(x'') \, \xi_{x'}(x'')$

$$= \int dx'' \, \delta(x'' - x) \, \delta(x'' - x')$$

$$= \delta(x - x')$$

This motivates the choices of $\eta_p(x)$ and $\delta_{x'}(x)$:

$$\langle x | \xi \rangle = \int dx' \, \delta(x - x') \, \xi(x')$$

$$= \xi(x')$$

a function of x

It's easy to show that $\xi(x')$ and $\tilde{\xi}(p)$ are Fourier Transforms of one another

&

are the definitions of the coordinate space

&

momentum space

WAVEFUNCTIONS

$\langle x | \xi \rangle$ & $\langle p | \xi \rangle$

This is the essence of Dirac's Representation Theory! he showed that

Heisenberg's discrete picture $\langle E_n | \xi \rangle$ was equivalent — just a different representation — to Schrödinger's continuous picture $\langle x | \xi \rangle$

Standard stuff in advanced quantum mechanics texts like Messiah & Cohen-Tannoudji, Diu, and Laloë

OPERATORS

State vectors are transformed in 3 ways:

1) Dynamics ... continuously. & understood

2) Measurement ... discretely & not understood

3) By symmetry operations

1) and 3) are maps:

$$|\xi\rangle \xrightarrow{M} |\alpha\rangle$$

$\downarrow$ through an operator

$$M|\xi\rangle = |\alpha\rangle$$

XII. M is linear if A. $M\{|\alpha\rangle \oplus |\beta\rangle\} = M|\alpha\rangle \oplus M|\beta\rangle$

B. For $a \in \mathbb{C}$,

$$M|a\xi\rangle = M|\xi\rangle a$$

XIII. M is self-adjoint or Hermitian if

$$M = M^{+} \quad \text{where} \quad (M^{+})^{i}{}_{j} = (M^{j}{}_{i})^{*}$$

IX. If the following relationship exists

$$A\,|\alpha_{(i)}\rangle = |\alpha_{(i)}\rangle\, a(i)$$

eigenstate

eigenvalue associated with i^{th} eigenstate

2 cases:

i) Two vectors expanded in terms of same basis set:

$$|\alpha\rangle = |\eta_{(i)}\rangle \alpha^{i}_{[\eta]}$$

$$|\beta\rangle = |\eta_{(j)}\rangle \beta^{j}_{[\eta]}$$

representatives of $|\alpha\rangle$ & $|\beta\rangle$ in basis $|\eta\rangle$

connected by $\quad |\alpha\rangle \to |\beta\rangle = M|\alpha\rangle$

$$= M\,|\eta_i\rangle \alpha^{i}_{[\eta]}$$

$$|\eta_j\rangle \beta^{j}_{[\eta]} =$$

Form scalar product

$$\langle \eta^{k}|\beta\rangle = \langle \eta^{k}|M|\eta_i\rangle \alpha^{i}_{[\eta]}$$

$$\downarrow$$

$$\langle \eta^{k}|\eta_j\rangle \beta^{j}_{[\eta]} = \delta^{k}{}_{j}\beta^{j}_{[\eta]} = \beta^{k}_{[\eta]}$$

Define the matrix element

$$\langle \eta^k | M | \eta_i \rangle \equiv M^k{}_i$$

$$\underset{\text{abstract}}{\uparrow} \qquad\qquad \underset{\text{a "representative" of } M \text{ in } |\eta\rangle \text{ basis}}{\uparrow}$$

iii) same vector, different bases

$$|\alpha\rangle = |\eta_i\rangle \alpha^i_{[\eta]} = |\gamma_m\rangle \alpha^m_{[\gamma]}$$

where <u>bases</u> are connected

$$|\eta\rangle \to |\gamma\rangle = M|\eta\rangle$$

or
$$|\gamma_m\rangle = |\eta_i\rangle M^i{}_m$$

Form inner product

$$\langle \eta^k | \alpha \rangle = \langle \eta^k | \eta_i \rangle \alpha^i_{[\eta]} = \langle \eta^k | \gamma_m \rangle \alpha^m_{[\gamma]}$$

$$\| \qquad\qquad\qquad \downarrow$$

$$\delta^k{}_i \alpha^i_{[\eta]} = \langle \eta^k | \eta_i \rangle M^i{}_m \alpha^m_{[\gamma]}$$

$$\downarrow$$

$$\delta^k{}_i$$

$$\alpha^k_{[\eta]} = M^k{}_m \alpha^m_{[\gamma]}$$

XV. Products of operators yield other operators

$$A|\eta_i\rangle = |\eta_j\rangle A^j{}_i$$

$$B|\eta_n\rangle = |\eta_k\rangle B^k{}_n$$

$$AB|\eta_n\rangle = A|\eta_k\rangle B^k{}_n = |\eta_j\rangle \underbrace{A^j{}_k B^k{}_n}$$

nicely arranged for matrix multiplication

$$= |\eta_j\rangle (\mathbb{A}\mathbb{B})^j{}_n$$

note $\quad BA|\eta_n\rangle = |\eta_m\rangle (\mathbb{B}\mathbb{A})^m{}_n$

which may not equal

The inverse must exist.

$$A|\xi\rangle = |\gamma\rangle$$

$$B|\gamma\rangle = |\xi\rangle$$

$$AB|\gamma\rangle = A|\xi\rangle = |\gamma\rangle \Rightarrow \quad \mathbb{A}\mathbb{B} = \mathbb{1}$$

$$\mathbb{A} = \mathbb{B}^{-1}$$

Again, consider a change of basis -

$$|\gamma_i\rangle = |\eta_k\rangle M^k{}_i \qquad\qquad |\eta\rangle \to |\gamma\rangle = M|\eta\rangle$$

$$|\eta_i\rangle = |\gamma_j\rangle (M^{-1})^j{}_i$$

and operate on one

$$A|\gamma_i\rangle = |\gamma_n\rangle A^n_{[\gamma]\,i}$$ ← matrix representative of A in basis γ

and the other

$$A|\eta_m\rangle = |\eta_p\rangle A^p_{[\eta]\,m}$$

How are they related?

$$A|\gamma_i\rangle = |\gamma_n\rangle A^n_{[\gamma]\,i} = |\eta_k\rangle M^k{}_n A^n_{[\gamma]\,i}$$

and $\quad A|\gamma_i\rangle = A|\eta_j\rangle M^j{}_i$

$$= |\eta_p\rangle A^p_{[\eta]\,j} M^j{}_i$$

so, $\quad |\eta_k\rangle M^k{}_n A^n_{[\gamma]\,i} = |\eta_p\rangle A^p_{[\eta]\,j} M^j{}_i$

since k is a summation index and dummy

For a given term labeled by k

$$M^k{}_n A^n{}_{[\gamma]i} = A^k{}_{[\eta]j} M^j{}_i$$

The bases are orthonormal, so,

$$(M^{-1})^l{}_k M^k{}_n = \delta^l{}_n \quad \text{so,}$$

$$\delta^l{}_n A^n{}_{[\gamma]i} = (M^{-1})^l{}_k A^k{}_{[\eta]j} M^j{}_i$$

$$A^l{}_{[\gamma]i} =$$

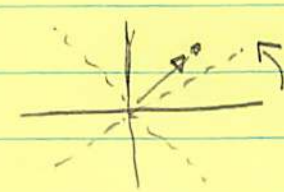
or,

$$\mathbb{A}_\gamma = \mathbb{M}^{-1} \mathbb{A}_\eta \mathbb{M} = \mathbb{M}^\dagger \mathbb{A}_\eta \mathbb{M}$$

representatives of A in each basis

took one basis into the other

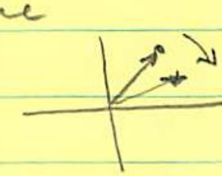
passive



Similarity Transformation
- determinant and trace are the same.

There is the other sense in which you rotate the vector and leave the basis alone

active



Suppose we have

$$|\alpha\rangle \rightarrow |\alpha'\rangle \quad \text{through} \quad |\alpha'\rangle = M|\alpha\rangle$$

and it's related to another vector by

$$|\beta\rangle = A|\alpha\rangle$$

How does this relationship look for the transformed vectors

$$|\alpha'\rangle = M|\alpha\rangle \Rightarrow |\alpha\rangle = M^{-1}|\alpha'\rangle$$

and

$$|\beta'\rangle = M|\beta\rangle$$

$$|\beta'\rangle = M|\beta\rangle = MA|\alpha\rangle$$

$$|\beta'\rangle = \underbrace{MAM^{-1}}_{A'}|\alpha'\rangle$$

so

$$A' = MAM^{-1}$$

the opposite

BACK TO GROUP THEORY!

DEF. Representation; A representation of a group $\mathcal{G}$ is a mapping of the elements of $\mathcal{G}$ onto a group of Linear Operators defined in a linear vector space, V.

$$\mathcal{G} = \{e, a, b, \ldots\}$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\mathcal{G} = \{\Gamma(e), \Gamma(a), \Gamma(b) \ldots\}$$

a "faithful" map

So. If we had

$$a \circ b = c \Rightarrow \Gamma(a) \otimes \Gamma(b) = \Gamma(c)$$

under what ever multiplication rule is operative for V.

Now, to D_3, A mathematical realization of the Δ — the basis vectors on which the Γ act which span V.

$$|x\rangle \longrightarrow |x'\rangle = \Gamma\,|x\rangle \qquad \text{abstract.}$$

Need a basis choice! $\quad |x'\rangle = |e'_{(j)}\rangle x^{j'}$

$$|x\rangle = |e_{(i)}\rangle x^{i}$$

from $\quad |x'\rangle = \Gamma |x\rangle$

$$= \Gamma |e_{(i)}\rangle x^i = |e'_{(j)}\rangle x^{j'}$$

$$\downarrow$$

$$= |e_{(j)}\rangle \Gamma^j{}_i x^i$$

If the two basis sets are the same, then the Γ are square..

$$|x'\rangle = |e_{(j)}\rangle x^{j'} = |e_{(j)}\rangle \Gamma^j{}_i x^i$$

and the components transform as

$$x^{j'} = \Gamma^j{}_i x^i$$

which begs for a matrix representation

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \Gamma^1{}_1 & \Gamma^1{}_2 & \cdots \\ \Gamma^2{}_1 & & \\ \vdots & & \\ \vdots & & \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ \vdots \end{pmatrix}$$

The component row-column rep is different from the bases'

$$|x\rangle = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \end{pmatrix}$$

$$|e\rangle = (\,|e_1\rangle, |e_2\rangle, \ldots\,)$$

Then $\quad |x\rangle = |e_{(i)}\rangle x^i \quad$ can be written

$$= (\,|e_1\rangle \;\; |e_2\rangle \;\; |e_3\rangle\,)\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$= |e_1\rangle x^1 + |e_2\rangle x^2 + |e_3\rangle x^3$$

But, the Γ can also transform the bases -

Bases transformation

$$|e_i\rangle \longrightarrow |e_i'\rangle = \Gamma |e_i\rangle = |e_j\rangle \Gamma^j{}_i$$

So the original transformation on vector components

$$x^{j\prime} = \Gamma^j{}_i x^i$$

Notice how the off diagonal terms are different from maybe what you expect -

$$|e_1'\rangle = |e_1\rangle \Gamma^1{}_1 + |e_2\rangle \Gamma^2{}_1$$

$$|e_2'\rangle = |e_1\rangle \Gamma^1{}_2 + |e_2\rangle \Gamma^2{}_2$$

where

$$x_1' = \Gamma^1{}_1 x^1 + \Gamma^1{}_2 x^2$$

$$x_2' = \Gamma^2{}_1 x^1 + \Gamma^2{}_2 x^2$$

Difference between and rotations.

active passive

Can get the elements by forming inner products.
In general.

$$|e'\rangle = \Gamma(g)\,|e\rangle$$

$$\langle e|e'\rangle = \langle e|\Gamma(g)|e\rangle$$

with the indices:

$$|e_i'\rangle = |e_j\rangle\,\Gamma^j{}_i$$

$$\langle e^k|e_i'\rangle = \langle e^k|e_j\rangle\,\Gamma^j{}_i$$

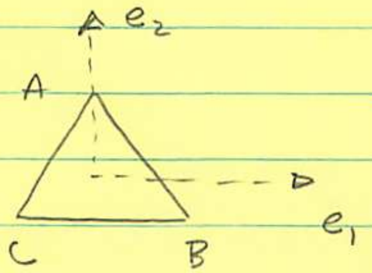
$$= \delta^k{}_j\,\Gamma^j{}_i = \Gamma^k{}_i$$

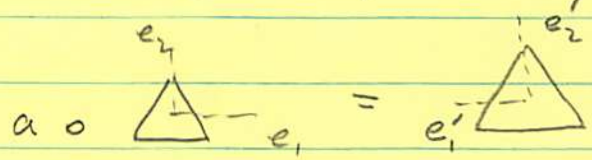
$$= \langle e^k|\Gamma(g)|e_i\rangle$$

so

$$\Gamma^k{}_i(g) = \langle e^k|\Gamma(g)|e_i\rangle$$

Okay. Back to D_3 → operate on coordinate system.





→ Tells me:
$\langle e|\Gamma(a)|e\rangle$

or. $e_1 \to e_1' = -e_1$, $e_2 \to e_2' = e_2$ $\Big\}$ $\Gamma(a)\, e_1 = -e_1$, $\Gamma(a)\, e_2 = e_2$

Calculate them.

$$\Gamma(a)^1{}_1 = \langle e^1 | \Gamma(a) | e_1 \rangle = -\langle e^1 | e_1 \rangle = -1$$

$$\Gamma(a)^1{}_2 = \langle e^1 | \Gamma(a) | e_2 \rangle = \langle e^1 | e_2 \rangle = 0 = \Gamma(a)^2{}_1$$

$$\Gamma(a)^2{}_2 = \langle e^2 | \Gamma(a) | e_2 \rangle = \langle e^2 | e_2 \rangle = +1$$

So, $$\Gamma^{(2)}(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

a 2-dimensional representation of P_3

So, $$|e'\rangle = \Gamma(a)|e\rangle$$

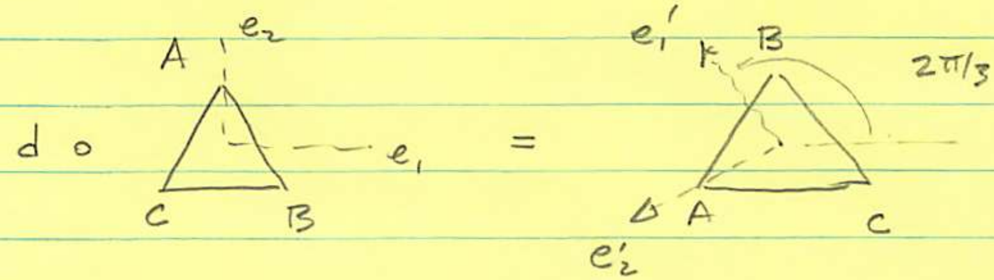
$$\Downarrow$$

$$|e_i'\rangle = |e_j\rangle \Gamma^j{}_i(a)$$

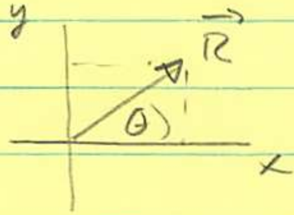
$$= (e_1, e_2)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= (-e_1, e_2) = (e_1', e_2')$$

How about d:



For an arbitrary vector rotation



$$\vec{R} = |R| \cos\theta \, \hat{i} + |R| \sin\theta \, \hat{j}$$

For our coordinate bases,

$$|e_i| = 1$$

$$\hat{i} = \hat{e}_1 \qquad \hat{j} = \hat{e}_2$$

So,
$$\hat{e}_1' = \hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta$$

$$\hat{e}_2' = -\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta.$$

For $d \rightarrow \theta = 2\pi/3$

$$\Gamma(d)'_1 = \langle e' | \Gamma(d) | e_1 \rangle$$

$$\Gamma(d) | e_1 \rangle = | e_1' \rangle$$

So,
$$\Gamma(d)'_1 = \langle e' | e_1 \cos\theta + e_2 \sin\theta \rangle$$

$$= \langle e' | e_1 \rangle \cos\theta \longrightarrow \cos\frac{2\pi}{3} = -1/2$$

$$\Gamma(d)^2_1 = \langle e^2 | \Gamma(d) | e_1 \rangle$$
$$= \langle e^2 | e_1 \cos\theta + e_2 \sin\theta \rangle$$
$$= \langle e^2 | e_2 \rangle \sin\theta = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

likewise $\quad \Gamma(d)^1{}_2 = -\frac{\sqrt{3}}{2}$

$$\Gamma(d)^2{}_2 = -1/2$$

$$\Gamma^i{}_j = \begin{pmatrix} \Gamma^1{}_1 & \Gamma^1{}_2 \\ \Gamma^2{}_1 & \Gamma^2{}_2 \end{pmatrix}$$

row

column

$$\Gamma(d) = \begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}$$

the others come from similar calculations.

So, we can fill out the rest of these

$\Gamma^{(3)}$: $\quad \Gamma^{(3)}(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma^{(3)}(d) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

$\Gamma^{(3)}(b) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \qquad \Gamma^{(3)}(f) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

$\Gamma^{(3)}(c) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

↑

all obey the D_3 multiplication table.

But, there are others.

$$\Gamma^{(1)}(a) = \Gamma^{(1)}(e) = \Gamma^{(1)}(b) = \Gamma^{(1)}(c) = \Gamma^{(1)}(d) = \Gamma^{(1)}(f) = 1$$

a one dimensional representation, $\Gamma^{(1)}$

and

$$\Gamma^{(2)}: \quad \Gamma^{(2)}(e) = \Gamma^{(2)}(d) = \Gamma^{(2)}(f) = 1$$

$$\Gamma^{(2)}(a) = \Gamma^{(2)}(b) = \Gamma^{(2)}(c) = -1$$

The D_3 group elements have names that differ by convention.

mine	other	still another
a	2_B	P
b	2_A	Q
c	2_C	R
d	3	C_3
f	3^2	$C_{\bar{3}}$
e	1	E

The representations are also named.

$\Gamma^{(1)}$: A_1 $\qquad$ $\Gamma^{(3)}$: E (!)

$\Gamma^{(2)}$: A_2

Remember the idea of group classes –

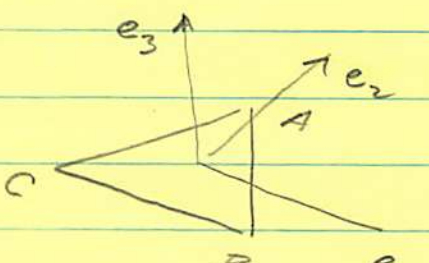
elements conjugate to one another?

I identified 3 D_3 classes.

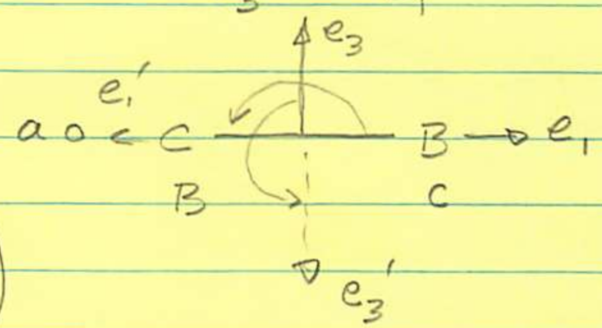
$\mathcal{C}_1$:	e	C_1	E
$\mathcal{C}_2$:	d, f	$2C_3$	$2y$
$\mathcal{C}_3$:	a, b, c	$3C_2, 3C_2'$	$3z$

terminology for some group elements

Suppose we had worked in 3 dimensions



then the, a, b, c elements would change the sign of e_3 & e_1



or $$\Gamma^{(4)}(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So, get a whole family of these

$$\Gamma^{(4)}(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \Gamma^{(4)}(d) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma^{(4)}(b) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \Gamma^{(4)}(f) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma^{(4)}(c) = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Notice the following: $\Gamma^{(4)}(g) = \left(\begin{array}{c|c} \Gamma^{(3)} & 0 \\ \hline 0 & \Gamma^{(2)} \end{array}\right)$

symbolically we would write

$$\Gamma^{(4)}(g_i) = \left(\begin{array}{c|c} \Gamma^{(3)}(g_i) & 0 \\ \hline 0 & \Gamma^{(2)}(g_i) \end{array}\right)$$

$$= \Gamma^{(3)}(g_i) \oplus \Gamma^{(2)}(g_i)$$