Lecture 5 What we did last time

- Finished Vector Spaces (yay!) and reminded you of Quantum Mechanics and "representation theory" as invented by Dirac and von Neumann.

- Derived the condition for a change of basis and the result on an operator

$$\mathbb{A}_\gamma = \mathbb{M}^{-1}\mathbb{A}_\eta \mathbb{M}$$

rep. of some operator A in 2 bases $|\gamma\rangle$ & $|\eta\rangle$ and the operator that maps them

$$|\eta\rangle \rightarrow |\gamma\rangle = M|\eta\rangle$$

→ a transformation of the <u>bases</u>, not the vectors

"passive rotation"

- Defined <u>The Representation</u> as a map of the elements of a $\mathcal{G}$ onto a $\mathcal{V}$

$$\{a, b, \ldots\} \longrightarrow \{\Gamma(a), \Gamma(b), \ldots\}$$

For bases $|e\rangle$ ... $\quad |x\rangle = |e_{(i)}\rangle x^i$

$$|x\rangle = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \end{pmatrix}$$

while

$$|e\rangle = \left(|e_{(1)}\rangle, |e_{(2)}\rangle \ldots \right)$$

The transformation is then

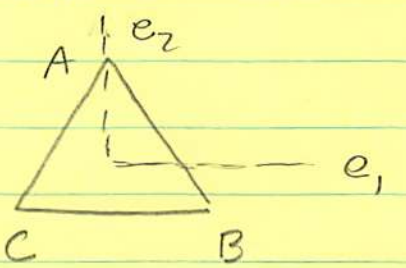
$$|e_{(i)}\rangle \longrightarrow |e'_{(i)}\rangle = \Gamma |e_{(i)}\rangle = |e_{(j)}\rangle \Gamma^j{}_i$$

where

$$|x\rangle \longrightarrow |e_{(j)}\rangle x^{j'} = |e_{(j)}\rangle \Gamma^j{}_i x^i$$

or $\quad x^{j'} = \Gamma^j{}_i x^i$

- Worked out the basis set transformations for D_3 brute force



a 2d representation

called $\Gamma^{(3)}(\gamma)$ $\quad \gamma = e, a, b, c, d, f$

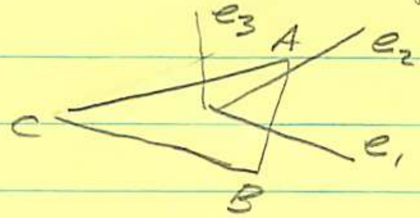
Also called by spectroscopists $\Gamma^{(E)}(\gamma)$

- Found 2 1d representations

$\Gamma^{(1)}$ and $\Gamma^{(2)}$ <u>or</u> $\Gamma^{(A_1)}$ and $\Gamma^{(A_2)}$

- Worked out a 3d representation of D_3

$\Gamma^{(4)}(\gamma_h)$ ...



But found that
it had the structure

$$\Gamma^{(4)}(\gamma_h) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\Gamma^{(E)}(\gamma)} & 0 \\ & & 0 \\ \hline 0 & 0 & \Gamma^{(A_2)} \end{array}\right]$$ Block Diagonal

$$= \Gamma^{(E)}(\gamma_h) \oplus \Gamma^{(A_2)}(\gamma_h)$$

DEF. <u>REDUCIBLE</u> <u>REPRESENTATION</u>: If it is possible to put a matrix representation into block diagonal form, it is a <u>reducible representation</u>. If not, it is irreducible. $\Gamma^{(red)}$

There are an ∞ number of representations for the point groups – but only a few irreducible representations and amazing similarities among all representations. $\boxed{\Rightarrow \text{there are as many IRR as \# classes}}$

DEF. <u>EQUIVALENT</u> <u>REPRESENTATION</u>: Two representations are said to be equivalent if two matrices $\Gamma(g)$ & $\Gamma'(g)$ are related – same dimension each – by

$$\Gamma'(g_i) = M^{-1}\Gamma(g_i)M$$

ie, a similarity transformation

same M for all group elements.

An infinite number of equivalent representations can be found.

Are there any characteristics that <u>don't</u> change under such a transformation?

Yes: remember that the trace of a matrix is unaltered by a similarity transformation -

But, one more transformation.

Suppose we have some rep $\Gamma(g)$. One can find a $\mathbb{D}$ such that

$$\Gamma^{red}(g_i) = \mathbb{D}^{-1}\,\Gamma(g_i)\,\mathbb{D}$$

same for each group element

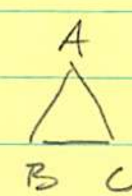
Remember that D_3 is also just a substitution group.

Here's a different basis: $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$$\Gamma_{D_3}(g)\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} A' \\ B' \\ C' \end{pmatrix}$$

whole other 3d rep.

Build from $a \circ$ 

$=$ 

so,
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} A \\ C \\ B \end{pmatrix}$$

Likewise...

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$c = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

call this $\Gamma^{(5)}$ .. notice: not block diagonal.

But,

$$\mathbb{D} = \begin{pmatrix} 0 & -2 & 0 \\ -\sqrt{3} & 1 & 0 \\ \sqrt{3} & 1 & 0 \end{pmatrix}$$

does it!

$$\Gamma^{(4)}(g_i) = \mathbb{D}^{-1}\,\Gamma^{(5)}(g_i)\,\mathbb{D}$$

In general, can get

$$\Gamma^{(red)}(g_k) = \left(\begin{array}{c|c|c|c} \Gamma^{(i)}(g_k) & 0 & 0 & 0 \\ \hline 0 & \Gamma^{(j)}(g_k) & & \\ 0 & & \Gamma^{(l)}(g_k) & \\ 0 & & & \end{array}\right)$$

Remember

$g \equiv$ order of a group, the # elements

$d_i \equiv$ dimensionality of the i^{th} IRR.

"irreducible representation"

<u>Important</u> orthogonality relation:

$$\sum_{p=1}^{g} \Gamma^{(i)\dagger}(\gamma_p)^{l}{}_{k}\, \Gamma^{(j)}(\gamma_p)^{m}{}_{n} = \delta^{i}{}_{j}\, \delta^{k}{}_{n}\, \delta^{m}{}_{l}\, \frac{g}{d_i}$$

matrix <u>elements</u>
just numbers

$$\Gamma^{(i)\dagger}(\gamma_p)^{k}{}_{l} = \left[\Gamma^{(i)}(\gamma_p)^{l}{}_{k}\right]^*$$

Ways to get non-zero

1. same IRR $\Rightarrow$ $i = j$

2. same row element $\Rightarrow$ $k = m$

3. same column element $\Rightarrow$ $l = n$

For our 3 IRR of D_3

$$\begin{array}{lll} & & e \quad a \quad b \quad c \quad d \quad f \\ A_1 \; \& \; A_1 & \sum_{p=1}^{6} \Gamma^{(1)*}(\gamma_p)\Gamma^{(1)}(\gamma_p) = & 1 + 1 + 1 + 1 + 1 + 1 = 6 \end{array}$$

$$= \frac{6}{1} \delta^1{}_1 \delta^1{}_1 \delta^1{}_1 = 6 \checkmark$$

A_1 & A_2 $$\sum_{p=1}^{6} \Gamma^{(1)*}(\gamma_p)\Gamma^{(2)}(\gamma_p) = 1 - 1 - 1 - 1 + 1 + 1 = 0$$

$$= \frac{6}{1} \delta^1{}_2 \delta^1{}_1 \delta^1{}_1 = 0 \checkmark$$

A_3 & A_2 $$\sum_{p=1}^{6} \Gamma^{(3)\dagger}(\gamma_p)^h{}_l \Gamma^{(2)}(\gamma_p)^m{}_n$$

$m = n = 1$ are only ones for 1d IRR

so make $h = l = 1$; $\delta^h{}_n \delta^m{}_l$

$$\sum_{p=1}^{6} \left[\Gamma^{(3)}(\gamma_p)\right]^1{}_1 \Gamma^{(2)}(\gamma_p)^1{}_1 =$$

$$1 + (1)(-1) + (-1/2)(-1) + (-1/2)(1) + (-1/2)(1) = 0$$

$$= \frac{6}{3} \delta^3{}_2 \delta^1{}_1 \delta^1{}_1 = 0 \checkmark$$

and so on.

Since all IRR are related by a similarity transformation (= equivalent), the TRACE is an important distinguishing feature

DEF, <u>CHARACTER</u>, $\chi^{(i)}(\gamma_p) \equiv$ trace of $i^{\underline{th}}$ IRR for group element γ_p

Remember, elements of a class are related by a similarity transformation,

the characters for all members of a given class are the same

$$\chi^{(i)}(\gamma_p) = Tr\left[\Gamma^{(i)}(\gamma_p)\right]$$

$$= \sum_{n=1}^{d_i} \Gamma^{(i)}(\gamma_p)^n{}_n$$

From orthogonality

$$\sum_{p=1}^{g} \left(\Gamma^{(i)}(\gamma_p)^h{}_h\right)^* \Gamma^{(j)}(\gamma_p)^n{}_n = \frac{g}{d_i}\,\delta^i{}_j\,\delta^h{}_n \quad \text{no } \underline{\text{sums}}$$

<u>now</u> do the traces $\Rightarrow \sum_h \sum_n$

$$\sum_{p=1}^{g} \chi^{(i)}(\gamma_p)^* \chi^{(j)}(\gamma_p) = \frac{g}{d_i} \delta^i{}_j \underbrace{\sum_h \sum_n \delta^h{}_n}_{d_i}$$

$$= g\, \delta^i{}_j$$

Now break the $\sum_{\gamma} \rightarrow \sum_{\mathcal{C}}$

group elements   classes

Define $N_r \equiv$ # elements in the r^{th} class

$$\sum_r \chi^{(i)}(\mathcal{C}_r)^* \chi^{(j)}(\mathcal{C}_r) N_r = \delta^i{}_j\, g$$

"row orthogonality"

This allows us to calculate and create the "character table" for a groups.

$\star$ → all that's needed for practical calculations $\star$

Suppose γ_a and γ_b are in the same class, so

$$\gamma_a = \gamma_m \gamma_b \gamma_m^{-1}$$

For any representation

$$\chi(\gamma_a) = \sum_i \Gamma(\gamma_a)^i{}_i = \sum_i \Gamma(\gamma_m \gamma_b \gamma_m^{-1})^i{}_i$$

problem

$$= \sum_i \sum_j \sum_h \Gamma(\gamma_m)^i{}_j \, \Gamma(\gamma_b)^j{}_h \, \Gamma(\gamma_m^{-1})^h{}_i$$

$$= \sum_j \sum_h \Gamma(\gamma_b)^j{}_h \, \Gamma(\gamma_m^{-1}\gamma_m)^h{}_j$$

$$= \sum_j \Gamma(\gamma_b)^j{}_j = \chi(\gamma_b)$$

These are presented this way:

$$\begin{array}{cc} & \xrightarrow{\text{class}} \\ \text{IRR} \downarrow & \begin{bmatrix} \quad \chi\text{'s} \quad \end{bmatrix} \end{array}$$

For D_3:

		(e) C_1	(d, f) $2C_3$	(a, b, c) $3C_2$
A_1	$\Gamma^{(1)}$	1	1	1
A_2	$\Gamma^{(2)}$	1	1	-1
E	$\Gamma^{(3)}$	2	-1	0

Another "orthogonality" relation

$$\sum_i \chi^{(i)}(\mathcal{C}_r)^* \chi^{(i)}(\mathcal{C}_s) = \delta^r{}_s \, \frac{g}{N_r}$$ "column orthogonality"

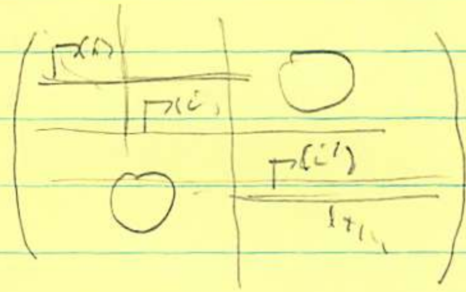
notice

class $\longrightarrow$ row orthogonality works.

IRR $\downarrow$ column orthogonality works

Suppose we have some $\Gamma^{(red)}$ — buried inside somewhere are some of the IRRs!

Symbolically,



$$\Gamma^{(red)}(\gamma_p) = \sum_i n_{[i]} \Gamma^{(i)}(\gamma_p)$$

times the i^{th} IRR appears.

$$\Gamma^{(red)}(\gamma_p) = n_1 \Gamma^{(1)}(\gamma_p) \oplus n_2 \Gamma^{(2)}(\gamma_p) \oplus \dots$$ ← physics! in this Direct Sum!

The trace can be easily written

$$\chi^{(red)}(\gamma_p) = \sum_i n_i \chi^{(i)}(\gamma_p)$$

Operate from left $\sum_p \chi^{(j)}(\gamma_p)^*$ & $\sum_i \leftrightarrow \sum_p$

$$\sum_p \chi^{(j)}(\gamma_p)^* \chi^{(red)}(\gamma_p) = \sum_i n_i \underbrace{\sum_p \chi^{(j)}(\gamma_p)^* \chi^{(i)}(\gamma_p)}_{g\delta^i_j \text{ (row orth.)}}$$

$$= n_j g$$

So, the frequency of the j^{th} IRR inside $\Gamma^{(red)}$ is

$$n_j = \frac{1}{g} \sum_p \chi^{(j)}(\gamma_p)^* \chi^{(red)}(\gamma_p)$$

$$= \frac{1}{g} \sum_r N_r \chi^{(j)}(\mathscr{C}_r) \chi^{(red)}(\mathscr{C}_r)$$

For example, take $\Gamma^{(5)}$ → how many times does $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}$ occur?

character table

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\Gamma^{(5)}$	3	0	1

$$n_1 = \frac{1}{6} \sum_{r=1}^{3} N_r \chi^{(1)}(\mathcal{C}_r)\, \chi^{(5)}(\mathcal{C}_r)$$

$$= \frac{1}{6}\left[1\cdot1\cdot3 + 2\cdot1\cdot0 + 3\cdot1\cdot1\right] = 1$$

likewise $n_3 = 1$, $n_2 = 0$

$$\Gamma^{(5)} = \Gamma^{(1)} \oplus \Gamma^{(3)} \qquad \text{no } \Gamma^{(2)}$$

The connection with Quantum Mechanics comes mostly through operations on functions – including basis functions

Define the action of operators on functions:

$$f(|x\rangle) \longrightarrow f'(|x\rangle) = \Gamma(g)\, f(|x\rangle) \qquad \text{abstract}$$

$$= f\left[\,|x_i\rangle \left(\Gamma^{-1}(g)\right)^i{}_j\,\right]$$

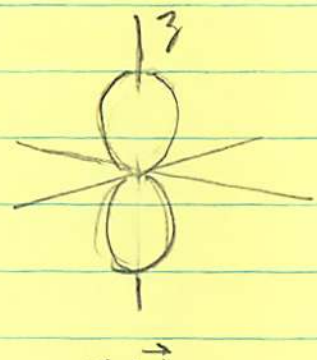
Complicated:

$f_i(|x\rangle)$ ← one vector space

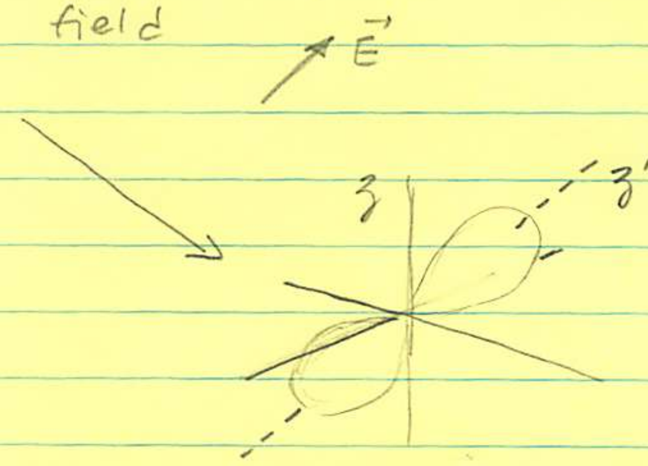
↑ another vector space

Here's a way to think about the presence of the Γ^{-1}

Suppose we have some wavefunction $\psi(x)$



and apply an electric field



notice that if we did both, rotate system and rotate coordinate axes, then get $\psi'(\vec{x}')$

BUT

$$\psi(x) = \psi'(x')$$

same function, just different labels

Here, $|x'\rangle = \Gamma(R)|x\rangle$

So, what we want is the new function in old coordinates.

$$\psi'(x') = \psi(\Gamma(R)^{-1}|x'\rangle)$$

dummy $\Rightarrow$ $\psi(x) = \psi(\Gamma(R)^{-1}x)$

This also takes into account the group multiplication table. So, if

$$a \circ b = c$$

then since Γ and $\gamma \in \{a, b, c \ldots\}$ are homomorphic

$$\Gamma(a)\,\Gamma(b) = \Gamma(c)$$

and this has to be preserved.

$$f \xrightarrow{a} f' \quad \Rightarrow \quad f'(|x\rangle) = f(\Gamma^{-1}(a)|x\rangle)$$

$$f' \xrightarrow{b} f'' \quad \Rightarrow \quad f''(|x\rangle) = f'(\Gamma(b)^{-1}|x\rangle)$$

so,

$$f''(|x\rangle) = f(\Gamma(b)^{-1}\,\Gamma(a)^{-1}|x\rangle)$$

$$= f((\underbrace{\Gamma(a)\,\Gamma(b)}_{\Gamma(c)})^{-1}|x\rangle)$$

$$f \xrightarrow{c = a \cdot b} f'' \qquad f''(|x\rangle) = f(\Gamma(c)^{-1}|x\rangle)$$

A function space is spanned by a set of basis functions. An operation on one can result in a linear combination of the others

$$\phi_i \rightarrow \phi_i' = \Gamma(g)\,\phi_i(|x\rangle)$$

$$= \phi_j\,\Gamma(g)^j{}_i$$

analogous of $|e_j\rangle$

So, transforming functions depends on how the components of the $|x\rangle$ transform $-\ \Gamma^{-1}$

So, we need $\Gamma^{-1}(g)|x\rangle$ for each group element. to see what happens to the function

As normal.

$$|e_i\rangle \rightarrow |e_i'\rangle = \Gamma|e_i\rangle$$

a vector: $|x\rangle = |e_l\rangle x^l$ original basis

$= |e_j'\rangle x'^j$ transformed basis: new components

$= \Gamma|e_j\rangle x^{j\prime}$ original basis – new components $\Rightarrow$ different function

Consider D_3 again

$$\Gamma(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \left(= \Gamma^{-1}(a) \right)$$

Need $\Gamma f(|x\rangle)$ by:

$$\Gamma^{-1}(a)\,|x\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^{1\prime} \\ x^{2\prime} \end{pmatrix} = \begin{pmatrix} -x^1 \\ x^2 \end{pmatrix}$$

So, functions of $|x\rangle$,

$$f(x,y) \rightarrow \Gamma(a) f(x,y) = f'(-x,y)$$

So, if $\quad f(x,y) = x^2 \longrightarrow f'(|x\rangle) = x^2$

How about d?

$$\Gamma(d) = \begin{pmatrix} -1/2 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -1/2 \end{pmatrix} \Rightarrow \Gamma^{-1}(d) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}$$

So,

$$\Gamma^{-1}(d)\,|x\rangle = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2\, x^1 + \sqrt{3}/2\, x^2 \\ -\sqrt{3}/2\, x^1 - 1/2\, x^2 \end{pmatrix}$$

So, if $f(x,y) = x^2$

$$f'(|x\rangle) = \tfrac{1}{4}x^2 + \tfrac{3}{4}y^2 - \frac{\sqrt{3}}{4}xy$$

Generally:

$$\Gamma^{-1}(e)\,|x\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Gamma^{-1}(a)\,|x\rangle = \begin{pmatrix} -x \\ y \end{pmatrix}$$

$$\Gamma^{-1}(b)\,|x\rangle = \begin{pmatrix} \tfrac{1}{2}x - \sqrt{3}/2\,y \\ -\frac{\sqrt{3}}{2}x - \tfrac{1}{2}y \end{pmatrix}$$

$$\Gamma^{-1}(c)\,|x\rangle = \begin{pmatrix} \tfrac{1}{2}x + \sqrt{3}/2\,y \\ \sqrt{3}/2\,x - \tfrac{1}{2}y \end{pmatrix}$$

$$\Gamma^{-1}(d)\,|x\rangle = \begin{pmatrix} -\tfrac{1}{2}x + \frac{\sqrt{3}}{2}y \\ -\frac{\sqrt{3}}{2} - \tfrac{1}{2}y \end{pmatrix}$$

$$\Gamma^{-1}(f)\,|x\rangle = \begin{pmatrix} -\tfrac{1}{2}x - \frac{\sqrt{3}}{2}y \\ \frac{\sqrt{3}}{2}x - \tfrac{1}{2}y \end{pmatrix}$$

Because, remember

$$f(|x\rangle) \rightarrow f'(|x\rangle) = \Gamma(g)\,f(|x\rangle)$$

$$f'(|x_j\rangle) = f\left(|x_i\rangle\,\Gamma^{-1}(g)^i{}_j\right)$$

Suppose we start with a basis function

$$f(|x\rangle) = y \equiv f_1$$

$$\left.\begin{aligned}
\Gamma(e) f_1 &= f_1 \\
\Gamma(a) f_1 &= y = f_1 \\
\Gamma(b) f_1 &= -\tfrac{\sqrt{3}}{2} x - \tfrac{1}{2} y \equiv f_3 \\
\Gamma(c) f_1 &= \tfrac{\sqrt{3}}{2} x - \tfrac{1}{2} y \equiv f_2 \\
\Gamma(d) f_1 &= -\tfrac{\sqrt{3}}{2} x - \tfrac{1}{2} y = f_3 \\
\Gamma(f) f_1 &= \tfrac{\sqrt{3}}{2} x - \tfrac{1}{2} y = f_2
\end{aligned}\right\} \text{ it } \underline{\text{closes}}$$

Notice also:

$$\Gamma(a) f_2 = \Gamma(a) \underbrace{\Gamma(c) f_1}_{f_2} = f_2\left((\Gamma(a)\Gamma(c))^{-1} |x\rangle \right)$$

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & +\sqrt{3}/2 \\ \frac{\sqrt{3}}{2} & -1/2 \end{pmatrix} \right]^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\frac{\sqrt{3}}{2} & -1/2 \end{pmatrix}}_{\text{which is } \Gamma^{-1}(d)} \begin{pmatrix} x \\ y \end{pmatrix}$$

so, $= \Gamma^{-1}(d) f_1 = f_3$

from which we can conclude $\Gamma(a)\Gamma(c) f_1 = f_3$

Also, for example, $\Gamma(a) f_3 = \Gamma(a)\Gamma(b) f_1$

$$= \Gamma(f) f_1 = f_2$$

We're working within a little functional space spanned by $f_1\ f_2\ f_3$ & on which a matrix representation of D_3 operates.

So: $$\Gamma(\gamma_k) f_i(|x\rangle) = f_j(|x\rangle) \Gamma(\gamma_k)^j{}_i$$

where we've found

$\Gamma(a) f_1 = f_1$ ✓
$\Gamma(a) f_2 = f_3$ ✓
$\Gamma(a) f_3 = f_2$ ✓

The Γ^5 representation of D_3 does this!

as bases $$(f_1\ f_2\ f_3) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\Gamma^{(5)}(a)} = (f_1\ f_3\ f_2)$$

The rest of $\Gamma^{(5)}$ also work – an appropriate representation for a set of basis functions that "start" with $f_1 = y$

All of the IRR have their own sets of basis functions.

$\Rightarrow$ basis functions can be classified according to the representation for which they form a basis

Let's keep track of the particular IRR

(1) $$\Gamma(\gamma_k) f_m^{(l)} = \sum_n f_n^{(l)} \Gamma^{(l)}(\gamma_k)^n{}_m$$

- the m^{th} function in the set of basis functions spanning the space for the l^{th} IRR.

- if the IRR is d-dimensional $\Rightarrow$ d basis functions (recall f_1, f_2, f_3), one for each column of the Γ matrix
  - say $f_m^{(l)}$ "belongs" to the m^{th} column of the l^{th} IRR

Want to find the others. $\longrightarrow$ a set of projection operators

Hit (1) by $\sum_k \Gamma^{(d)}(\gamma_k)^{\dagger i}{}_j$ from the left.

$$\sum_h \Gamma^{(d)}(\gamma_h)^{+j}{}_j \, \Gamma(\gamma_h) f^{(\ell)}_m = \sum_n f_n^{(\ell)} \underbrace{\sum_h \Gamma^{(d)}(\gamma_h)^{+i}{}_j \, \Gamma^{(\ell)}(\gamma_h)^n{}_m}_{\substack{\text{orthogonality} \\ \delta^d{}_\ell \, \delta^n{}_j \, \delta^i{}_m \frac{g}{d_\ell}}}$$

$$= f_j^{(\ell)} \, \delta^d{}_\ell \, \delta^i{}_m \, \frac{g}{d_\ell}$$

Define

$$\mathbb{P}^{(d)i}{}_j \equiv \frac{d_d}{g} \sum_h \Gamma^{(d)+}(\gamma_h)^i{}_j \, \Gamma(\gamma_h)$$

so that LHS =

$$\mathbb{P}^{(d)i}{}_j \, f_m^{(\ell)} = \delta^d{}_\ell \, \delta^i{}_m \, f_j^{(\ell)}$$

same IRR
different "column"

so, you shift from column to column, filtering out the functions .. like magic.

so $\quad f_j^{(\ell)} = \mathbb{P}^{(\ell)m}{}_j \, f_m^{(\ell)}$ ← same IRR ← different column

Here's the basic problem –

Suppose we have an arbitrary function F, how can we find the basis functions associated with a specific IRR?

$\downarrow$

there might be more than one, so we decompose the function into the contributing IRR's

set $m=j$ in $\mathbb{P}^{(l)m}{}_j$ – a particular situation

$$\mathbb{P}^{(l)m}{}_m = \frac{d_l}{g} \sum_h \Gamma^{(l)}(\gamma_h)^{+m}{}_m \, \Gamma(\gamma_h)$$

so, if it operates on an arbitrary function and

$$\mathbb{P}^{(l)m}{}_m \, \psi = \psi$$

then one can conclude that ψ belongs to the m^{th} column of the l^{th} IRR – can classify functions.

Acts like a projection operator.

$$P^{(l)}_m \equiv \mathbb{P}^{(l)m}{}_m = \frac{d_l}{g} \sum_h \Gamma^{(l)}(\gamma_h)^{+m}{}_m \, \Gamma(\gamma_h)$$

Claim: any arbitrary function in which $\Gamma(\gamma_k)$ can operate can be written as sum of a set of functions – all of which are basis functions of that IRR

$$\psi(x) = \sum_{l=1}^{N_l} \sum_{j=1}^{n_l} f_j^{(l)}(|x\rangle)$$

N_l: # IRR of group
n_l: dimensionality of l^{th} IRR

Now, "project":

$$P^{(l)}{}_m \psi(x) = \frac{d_l}{g} \sum_k \Gamma^{(l)}(\gamma_k)^{+\,m}{}_m \Gamma(\gamma_k) \psi(x)$$

$$= \sum_{d=1}^{N_d} \sum_{j=1}^{n_d} \mathbb{P}^{(l)\,m}{}_m f^{(d)}_j$$

$$= \sum_d \sum_j \delta^d{}_l \delta^m{}_j f^{(d)}{}_j$$

$$P^{(l)}{}_m \psi(x) = f^{(l)}{}_m \qquad \text{project out}$$

If $\sum_m$ and operate on $f^{(d)}{}_m$

$$\sum_m P_m^{(l)} f^{(d)}{}_m = \sum_m \mathbb{P}^{(l)\,m}{}_m f^{(d)}{}_m$$

$$= \frac{d_l}{g} \sum_k \underbrace{\sum_m \Gamma^{(l)}(\gamma_k)^{+\,m}{}_m}_{\chi} \Gamma(\gamma_k) f^{(d)}{}_m$$

$$\delta^l{}_d f_m^{(d)} = \frac{d_l}{g} \sum_k \chi^{*(l)}(\gamma_k) \Gamma(\gamma_k) f_m^{(d)}$$

so, we get another projection operator

$$P^{(\ell)} f_m^{(d)} = f_m^{(d)} \delta^{\ell}{}_{d}$$

So on an arbitrary function

$$P^{(\ell)} \psi = \frac{d_\ell}{g} \sum_k \chi^{*(\ell)}(\gamma_k) \Gamma(\gamma_k) \psi = f^{(\ell)}$$

Example $\psi(|x\rangle) = x^2$

Use the $P^{(\ell)}$'s to decompose ψ into components which transform irreducibly under each IRR of D_3

JUST NEED THE CHARACTER TABLE!

		(e) C_1	(d, f) $2C_3$	(a, b, c) $3C_2$
$\Gamma^{(1)}$	A_1	1	1	1
$\Gamma^{(2)}$	A_2	1	1	-1
$\Gamma^{(3)}$	E	2	-1	0

(from page 88)