Continuous Groups.

Suppose: A finite group $\mathcal{G}$ of order g consists of elements $\{g_1, g_2, \cdots g_g\}$

Group property: $g_k = g_i \circ g_j$

Fix j and vary i through the whole group $\Rightarrow$ k runs through the whole group as well.

Could be a <u>table</u>. But, also like a functional relationship:

$$g_k = f(g_i, g_k)$$

NOW - suppose the group element "variable" is allowed to be <u>continuous</u>.

$$g_i \longrightarrow g(\alpha)$$

like limit of discrete variable $\ni$ $g(\alpha)$ & $g(\alpha')$ are near if $|\alpha - \alpha'|$ small

"rotations" immediately come to mind.. but not only thing

Group: single parameter, α, -- can have ∞ values.

1. $g(\alpha) \circ g(\alpha') = g(\alpha'')$ "group property"

2. $g(\alpha) \circ [g(\alpha') \circ g(\alpha'')] = [g(\alpha) \circ g(\alpha')] \circ g(\alpha'')$

3. $\forall_{\alpha^0}$ such that $g(\alpha) \circ g(\alpha^0) = g(\alpha)$

4. $\forall_\alpha$, there is an $\tilde{\alpha}$, $\ni$ $g(\alpha) \circ g(\hat{\alpha}) = g(\alpha^0)$

The group property acts as a function

$$\alpha'' = f(\alpha, \alpha')$$

A one-parameter, continuous group.

There are usually multiple parameters.

α^σ or $\vec{\alpha}$ $\sigma = 1, 2 \ldots r$ r parameter, continuous group.

$$\alpha''^\sigma = \varphi^\sigma(\vec{\alpha}; \vec{\alpha}') = \varphi^\sigma(\alpha^1, \alpha^2, \alpha^3 \ldots \alpha^r; \alpha^{1\prime}, \alpha^{2\prime}, \ldots \alpha^{r\prime})$$

If φ^σ are differentiable to all orders (analytic) in both arguments $\rightarrow$ $\mathcal{G}$ is an "r-parameter Lie Group"

1st continuous group: rotations in 2 dimensions, R_2

Abelian:

$$0 < \alpha < 2\pi \qquad \text{and} \qquad \Gamma(\gamma) = \Gamma(\alpha)\Gamma(\beta)$$
$$\Rightarrow \gamma = \alpha + \beta.$$

IRR are 1 dimensional

Need to find $\Gamma(\alpha)$ such that

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta) \qquad \& \qquad \Gamma(0) = 1$$

Differentiate wrt β

$$\Gamma(\alpha)\Gamma'(\beta) = \Gamma'(\alpha+\beta)$$

let $\beta \to 0$ $\qquad \Gamma(\alpha)\Gamma'(0) = \Gamma'(\alpha)$

which has solutions $\qquad \Gamma(\alpha) = e^{\alpha\Gamma'(0)}$

So that it's continuous $\quad \Gamma(\alpha) = \Gamma(\alpha + 2\pi)$
then $\Gamma'(0)$ must be an imaginary integer

Define $\qquad m = i\Gamma'(0) \qquad\qquad m = 0, \pm 1, \pm 2 \ldots$

$$\Gamma(\alpha) = e^{-im\alpha}$$

Unitary and Orthogonal

$$\int_0^{2\pi} \Gamma^{(m)+}(\alpha)\, \Gamma^{(m')}(\alpha)\, d\alpha = 2\pi\, \delta^{mm'}$$

On basis vectors: — call it R

$$R_2(\alpha)\,|e_1\rangle = \cos\alpha\,|e_1\rangle + \sin\alpha\,|e_2\rangle$$

$$R_2(\alpha)\,|e_2\rangle = -\sin\alpha\,|e_1\rangle + \cos\alpha\,|e_2\rangle$$

so,

$$R_2(\alpha)\left[\,|e_1\rangle \pm i\,|e_2\rangle\right] = e^{\mp i\alpha}\left[\,|e_1\rangle \pm i\,|e_2\rangle\right]$$

then $\quad |\varphi_\pm\rangle \equiv |e_1\rangle \pm i\,|e_2\rangle \quad$ transform

according to the IRR of R_2 with $m = \pm 1$

Suppose polar coordinates

$$\psi(r,\theta) \rightarrow \psi'(r,\theta) = \Gamma[R_2(\theta)]\,\psi(r,\theta)$$

$$= \psi(r, R_2^{-1}\theta) = \psi(r, \theta - \alpha)$$

So, $\quad \psi(r,\theta) = \psi(r)e^{im\theta}$

will transform according to the $\Gamma^{(m)}(\alpha)$ IRR of R_2

could expand any function in terms of the basis functions of R_2

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} \psi_m(r) e^{im\theta}$$

which is a Fourier Series.

suppose $\quad x \to x+\alpha \qquad$ translations in 1d

$\mathcal{T}_1$

same everything: $\quad \Gamma^{(k)}(\alpha) = e^{-ik\alpha}$

BUT $\quad 0 < \alpha < \infty$ .. not periodic $\Rightarrow$ k not integer.

Technical Issue - For a group element γ,

γ depends on $\vec{\alpha} = \{\alpha^1, \alpha^2 .. \alpha^r\} \Rightarrow \gamma_{\vec{\alpha}}$

$$\Gamma(\gamma_{\vec{\alpha}})|\xi\rangle = |\xi'\rangle$$

choose a basis: $\quad \Gamma(\gamma_{\vec{\alpha}})|\eta_i\rangle \xi^i = |\eta_i\rangle \xi^{i'}$

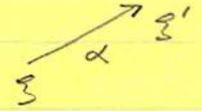
$$|\eta_i\rangle \Gamma(\gamma_{\vec{\alpha}})^i{}_k \xi^k = |\eta_i\rangle \xi^{i'}$$

so: $\quad \xi^i \to \xi^{i'} = \Gamma(\gamma_{\vec{\alpha}})^i{}_k \xi^k$

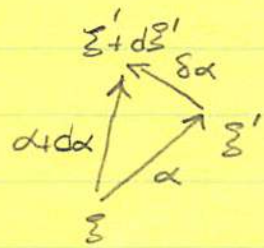
$$\xi^{i'} = F^i(\xi^1, \xi^2 .. \xi^n; \alpha^1, \alpha^2 .. \alpha^r)$$

So, the vector transforms according to a change in the parameter

α takes $\xi \rightarrow \xi'$



but infinitesimally close to ξ' is $\xi' + d\xi'$ which can be gotten from ξ' by $\delta\alpha$



But, can get there directly by the additive $\alpha + d\alpha$

So, two ways: $\quad F(\xi; \alpha + d\alpha) = \xi' + d\xi'$

$$F(\xi'; \delta\alpha) = \xi' + d\xi'$$

so,

$$d\xi' = F(\xi'; \delta\alpha) - \xi'$$

$$= F(\xi'; \delta\alpha) - F(\xi', 0)$$

or,

$$d\xi' = \left[\frac{\partial F}{\partial \alpha}(\xi; \alpha) \right]_{\alpha = 0} \delta\alpha \qquad \star$$

For a general case

$$d\xi^i = \sum_\sigma \left[\frac{\partial F^i(\vec{\xi}; \vec{\alpha})}{\partial \alpha^\sigma} \right]_{\vec{\alpha} = 0} \delta\alpha^\sigma$$

$$d\xi^i = u^i{}_\sigma(\vec{\xi})\, \delta\alpha^\sigma \qquad \text{vectors}$$

For the parameters.

$$g(\alpha) = g(\alpha') \circ g(\alpha'')$$

so $$\alpha = \varphi(\alpha'^1 \ldots \alpha'^r ; \alpha''^1 \ldots \alpha''^r)$$

for a set $$g(\vec{\alpha}) = g(\vec{\beta}) \circ g(\vec{\epsilon})$$

$$\alpha^\sigma = \varphi^\sigma(\vec{\beta} ; \vec{\epsilon})$$

which says again $$\alpha^\sigma + d\alpha^\sigma = \varphi^\sigma(\alpha^1 \ldots \alpha^r ; \delta\alpha^1 \ldots \delta\alpha^r)$$

$$d\alpha^\sigma = \sum_{\rho=1}^{r} \left[\frac{\partial \varphi^\sigma(\vec{\alpha} ; \vec{\beta})}{\partial \beta^\rho} \right]_{\beta=0} \delta\alpha^\rho$$

$$= V^\sigma{}_\rho(\vec{\alpha})\, \delta\alpha^\rho \qquad \text{so} \quad V^\sigma{}_\rho(0) = \delta^\sigma{}_\rho$$

The inverse exists: $$V^\sigma{}_\rho \Lambda^\rho{}_\tau = \delta^\sigma{}_\tau$$

so $$\Lambda^\rho{}_\tau\, d\alpha^\sigma = \Lambda^\rho{}_\tau V^\sigma{}_\rho\, \delta\alpha^\rho = \delta^\sigma{}_\tau\, \delta\alpha^\rho$$

so $$\Lambda^\rho{}_\tau\, d\alpha^\tau = \underline{\underline{\delta\alpha^\rho}}$$

So, the vector change is

$$d\xi^i = u^i{}_\sigma(\vec{\xi}) \Lambda^\sigma{}_\rho(\vec{\alpha})\, d\alpha^\rho$$

or

$$\frac{d\xi^i}{d\alpha^\rho} = u^i{}_\sigma(\vec{\xi}) \Lambda^\sigma{}_\rho(\vec{\alpha})$$

For an arbitrary function:

$$\Gamma(\gamma_\mu) f(\vec{\xi}) = f'(\vec{\xi}) = f(\vec{\xi}')$$

since $\quad f(\vec{\xi}') = f(\vec{\xi} + \delta\vec{\xi}) \xrightarrow[\text{limit}]{} \frac{\partial f(\vec{\xi})}{\partial \xi^i} d\xi^i + f(\vec{\xi})$

$$f(\vec{\xi}') - f(\vec{\xi}) = \Delta f(\vec{\xi}) = \frac{\partial f(\vec{\xi})}{\partial \xi^i} \underbrace{d\xi^i}_{u^i{}_\sigma \delta\alpha^\sigma} \equiv df(\xi)$$

since $\delta\xi \to 0$ as $\alpha \to 0$.

start from $\alpha = 0$ and move from there

$$df(\xi) = \frac{\partial f(\vec{\xi})}{\partial \xi^i} u^i{}_\sigma(\vec{\xi}) \delta\alpha^\sigma$$

$$df = \delta\alpha^\sigma u^i{}_\sigma(\vec{\xi}) \frac{\partial}{\partial \xi^i} f(\vec{\xi})$$

$$df \equiv \delta\alpha^\sigma X_\sigma f(\vec{\xi})$$

defining $\quad X_\sigma \equiv u^i{}_\sigma \frac{\partial}{\partial \xi^i} \quad$ infinitesimal generators

describes how coordinates infinitesimally change wrt infinitesimal changes in parameters

Suppose $\quad f(\vec{\xi}) = \xi^i$

$$\Gamma(\gamma) f = \Gamma(\gamma)\xi^i = \xi^{i\prime} = \xi^i + d\xi^i$$

$$= \xi^i + \delta\alpha^\sigma X_\sigma \xi^i$$
$$= \xi^i + u^j{}_\sigma \frac{\partial \xi^i}{\partial \xi^j}\delta\alpha^\sigma$$

$$= \xi^i + u^j{}_\sigma \delta^i{}_j \delta\alpha^\sigma$$
$$\xi^{i\prime} = \xi^i + \underbrace{u^i{}_\sigma \delta\alpha^\sigma}_{d\xi^i} \quad ✓$$

So, now consider a transformation infinitesimally close to the identity, from the above,

$$\Gamma(\gamma_{\vec{\alpha}}) = \mathbb{1} + \delta\alpha^\sigma X_\sigma$$

so

$$\Gamma(\gamma_{\vec{\alpha}}) f(\vec{\xi}) = f(\xi) + \delta\alpha^\sigma X_\sigma f(\xi)$$

$$= (\mathbb{1} + \delta\alpha^\sigma X_\sigma) f(\xi)$$

2 successive transformations

$$\Gamma(\gamma_{\vec{\alpha}})\Gamma(\gamma_{\vec{\beta}}) = (1 + \delta\alpha^\sigma X_\sigma)(1 + \delta\beta^\rho X_\rho)$$
$$= 1 + \delta\beta^\rho X_\rho + \delta\alpha^\sigma X_\sigma + \mathcal{O}(\delta\alpha\,\delta\beta)$$

$$= 1 + \delta\beta^1 X_1 + \delta\beta^2 X_2 + \cdots + \delta\alpha^1 X_1 + \delta\alpha^2 X_2 + \cdots$$
$$= 1 + (\underbrace{\delta\beta^\sigma + \delta\alpha^\sigma}_{\delta\epsilon^\sigma})X_\sigma \qquad \delta\epsilon^\sigma \text{ additive}$$

$$\Gamma(\gamma_{\vec{\alpha}})\,\Gamma(\beta_{\vec{\alpha}}) = 1 + \delta\epsilon^{\sigma} X_{\sigma} = \Gamma(\gamma_{\vec{\epsilon}})$$ group property.

Let α be a small β .. $\delta\beta$

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)$$
$$\Gamma(\delta\beta)\Gamma(\beta) = \Gamma(\delta\beta+\beta)$$

Now

$$(1 + \delta\beta^{\sigma} X_{\sigma})\,\Gamma(\beta) = \Gamma(\beta) + \delta\beta^{\sigma} X_{\sigma}\Gamma(\beta) = \Gamma(\beta + \delta\beta)$$

$$\frac{\Gamma(\beta + \delta\beta) - \Gamma(\beta)}{\delta\beta^{\sigma}} = X_{\sigma}\,\Gamma(\beta)$$

$\lim_{\delta\beta \to 0}$ $$\frac{d\Gamma(\vec{\beta})}{d\beta^{\sigma}} = X_{\sigma}\Gamma(\vec{\beta})$$

Formal solution: $$\Gamma(\vec{\beta}) = e^{X_{\sigma}\beta^{\sigma}}$$

Go back to $$\frac{\partial \xi^i}{\partial \alpha^\rho} = u^i{}_\sigma(\vec{\xi})\Lambda^\sigma{}_\rho(\vec{\alpha})$$

rate of change of ξ wrt α, from $\xi=0$ @ $\alpha=0$

Demand $$\frac{\partial^2 \xi^i}{\partial \alpha^\tau \partial \alpha^\rho} = \frac{\partial^2 \xi^i}{\partial \alpha^\rho \partial \alpha^\tau}$$

$$u^i{}_\sigma(\xi)\frac{\partial \Lambda^\sigma{}_\rho(\alpha)}{\partial \alpha^\tau} + \frac{\partial u^i{}_\sigma(\xi)}{\partial \alpha^\tau}\Lambda^\sigma{}_\rho(\alpha) = u^i{}_\sigma(\xi)\frac{\partial \Lambda^\sigma{}_\tau(\alpha)}{\partial x^\rho} + \frac{\partial u^i{}_\sigma}{\partial x^\rho}\Lambda^\sigma{}_\tau$$

$$\frac{\partial u^i{}_\sigma(\xi)}{\partial \alpha^\tau}\Lambda^\sigma{}_\rho(\alpha) = \frac{\partial u^i{}_\sigma(\xi)}{\partial \xi^j}\underbrace{\frac{\partial \xi^j}{\partial \alpha^\tau}}_{u^j{}_\nu \Lambda^\nu{}_\tau}\Lambda^\sigma{}_\rho(\alpha) \qquad \frac{\partial u^i{}_\sigma}{\partial x^\rho}\Lambda^\sigma{}_\tau = \frac{\partial u^i{}_\sigma}{\partial \xi^j}\underbrace{\frac{\partial \xi^j}{\partial \alpha^\rho}}_{u^j{}_\epsilon \Lambda^\epsilon{}_\rho}\Lambda^\sigma{}_\tau$$

rearrange

$$u^j{}_\nu \frac{\partial u^i{}_\sigma}{\partial \xi^j}\Lambda^\nu{}_\tau \Lambda^\sigma{}_\rho - u^j{}_\epsilon \frac{\partial u^i{}_\sigma}{\partial \xi^j}\Lambda^\epsilon{}_\rho \Lambda^\sigma{}_\tau = u^i{}_\sigma\left(-\frac{\partial \Lambda^\sigma{}_\rho}{\partial \alpha^\tau} + \frac{\partial \Lambda^\sigma{}_\tau}{\partial x^\rho}\right)$$

$V^\tau{}_\rho V^\rho{}_\eta \longrightarrow$

pair up $V\Lambda$'s and get lots of δ's

$$u^j{}_\nu \frac{\partial u^i{}_\sigma}{\partial \xi^j}\delta^\nu{}_\rho \delta^\sigma{}_\eta - u^j{}_\epsilon \frac{\partial u^i{}_\sigma}{\partial \xi^j}\delta^\epsilon{}_\eta \delta^\sigma{}_\rho = (\qquad) u^i{}_\sigma V^\tau{}_\rho V^\rho{}_\eta$$

$$u^j{}_\rho \frac{\partial u^i{}_\eta}{\partial \xi^j} - u^j{}_\eta \frac{\partial u^i{}_\rho}{\partial \xi^j} = \left(\frac{\partial \Lambda^\sigma{}_\tau}{\partial \alpha^\rho} - \frac{\partial \Lambda^\sigma{}_\rho}{\partial x^\tau}\right) V^\tau{}_\rho V^\rho{}_\eta u^i{}_\sigma$$
$$\equiv c^\sigma{}_{\rho\eta}\, u^i{}_\sigma$$

$\frac{\partial}{\partial \xi^i}$ from right $\longleftarrow$

$$u^j_\rho \frac{\partial}{\partial \xi^j} u^i_\eta \frac{\partial}{\partial \xi^i} - u^j_\eta \frac{\partial}{\partial \xi^j} u \frac{\partial}{\partial \xi^i} = c^\sigma_{\rho\eta} u^i_\sigma \frac{\partial}{\partial \xi^i}$$

recognize the infitesimal generators

$$X_\rho X_\eta - X_\eta X_\rho = c^\sigma_{\rho\eta} X_\sigma$$

$$[X_\rho, X_\eta] = c^\sigma_{\rho\eta} X_\sigma \qquad \text{Lie Algebra}$$

structure constants

Fundamental Theorem of Lie: EVERYTHING you want to know about a Lie Group is the <u>algebra</u> satisfied by the generators.

→ Don't need the multiplication table

$\boxed{\text{example 1}}$ $\mathcal{T}_1$ .. the 1 dimensional shift

$$\xi \to \xi' = a\xi + b$$

want Ø parameter values to ⇒ no change

Recast:

$$\xi \to \xi' = (1+\alpha_1)\xi + \alpha_2$$