Infinitesimal transformation:

$$\xi' = \xi + \delta\xi = (1+\delta\alpha_1)\xi + \delta\alpha_2$$
$$= \xi + \delta\alpha_1 \xi + \delta\alpha_2$$

$$\delta\xi = \delta\alpha_1 \xi + \delta\alpha_2$$

remember: $$\delta\xi^i = u^i{}_\sigma \delta\alpha^\sigma$$

1 dimension:

$$\delta\xi^1 = u^1{}_\sigma \delta\alpha^\sigma$$

$$= u^1{}_1 \delta\alpha^1 + u^1{}_2 \delta\alpha^2$$
$$= (\xi)\delta\alpha^1 + (1)\delta\alpha^2$$

so,

$$X_\sigma = u^i{}_\sigma \frac{\partial}{\partial\xi^i} \Rightarrow X_1 = u^1{}_1 \frac{\partial}{\partial\xi^1} = \xi^1 \frac{\partial}{\partial\xi^1}$$

$$X_2 = u^1{}_2 \frac{\partial}{\partial\xi^1} = \frac{\partial}{\partial\xi^1}$$

Operate on something:

$$[X_1, X_2]F = X_1 X_2 F - X_2 X_1 F$$
$$= \xi^1 \frac{\partial}{\partial\xi^1}\frac{\partial}{\partial\xi^1} F - \frac{\partial}{\partial\xi^1} \xi^1 \frac{\partial}{\partial\xi^1} F$$
$$= \xi \frac{\partial^2}{\partial\xi^2} F - \xi \frac{\partial^2}{\partial\xi} F - \frac{\partial}{\partial\xi^1} F$$
$$= -\frac{\partial}{\partial\xi^1} F = -X_2 F$$

So, $[X_1, X_2] = -X_2$ $\qquad C^1_{12} = 0 = C^1_{21}$

$\qquad C^2_{12} = -1 = -C^2_{21}$

$\boxed{\text{Example 2}}$

$$|\xi\rangle \longrightarrow |\xi'\rangle = R(\vec{\alpha})|\xi\rangle$$

in a basis $\qquad |e_i\rangle \xi^{i\prime} = |e_i\rangle R(\alpha)^i{}_j \xi^j$

so coordinates $\qquad \xi^{i\prime} = R(\alpha)^i{}_j \xi^j$

$\boxed{\text{Demand}}$ ① Unitary: $\qquad RR^+ = 1$

let $\quad A = \mathbb{R}^+ \quad \Rightarrow \quad \mathbb{R}A = \mathbb{1}$

$R^i{}_j A^j{}_k = \delta^i{}_k$

But $\quad A^j{}_k = R^{+j}{}_k = R^{*}_k{}^j = (R^k{}_j)^*$

so $\quad RR^+ = 1 \Rightarrow \sum_j R^i{}_j R^{+j}{}_k = \delta^i{}_k = \sum_j R^i{}_j R^{*}_k{}^j$

② Real

So, $\quad \sum_j R^i{}_j R_k{}^j = \delta^i{}_k$

$\quad \sum_j R^i{}_j (R^j{}_k)^T = \delta^i{}_k$

$$\mathbb{R}\mathbb{R}^T = \mathbb{1}$$

③ Length-preserving.

as usual $\quad |e_i'\rangle = |e_j\rangle R^j{}_i \quad \Rightarrow \langle e^{i'}| = R^{+i}{}_j \langle e^j|$

inverse $\quad |e_i\rangle = |e_j'\rangle S^j{}_i$

from
$$\langle e^{h'}|e_i\rangle = R^{+h}{}_l \langle e^l|e_i\rangle = R^{+h}{}_l \delta^l{}_i = R^{+h}{}_i$$

$$\langle e^{h'}|e_j'\rangle S^j{}_i = \delta^h{}_j S^j{}_i = S^h{}_i$$

so $\quad S^h{}_i = R^{+h}{}_i = R^*{}_i{}^h = (R^i{}_h)^*$

Bases are real _ matrices are real,

$$S^h{}_i = R^i{}_h \Rightarrow (R^{-1})^i{}_h = R^h{}_i \Rightarrow \mathbb{R}^{-1} = \mathbb{R}^T$$

Also,
$$\langle e^{h'}|e_i'\rangle = R^{+h}{}_l \langle e^l|e_j\rangle R^j{}_i = R^{+h}{}_l R^j{}_i \delta^l{}_j$$
$$\delta^h{}_i = R^{+h}{}_j R^j{}_i$$

$\Rightarrow \mathbb{R}$ is orthogonal $\Rightarrow \langle e^{h'}|e_i'\rangle = \langle e^h|e_i\rangle$

hence, length-preserving

from $R^{+h}{}_j R^j{}_i = \delta^h{}_i$ orthogonal

$R^{*}_{j}{}^{h} R^j{}_i =$

$(R^j{}_h)^* R^j{}_i =$

$R^j{}_h R^j{}_i = \delta^h{}_i$ real

$\det(R^j{}_h R^j{}_i) = \det \delta^h{}_i = 1 \Rightarrow h = i$

so $[\det(R^j{}_i)]^2 = 1 \Rightarrow \det \mathbb{R} = \pm 1$

so 2 different sets possible $\det \mathbb{R} = 1$ group of proper rotations
$\det \mathbb{R} = -1$ not a group

For infinitesimal rotations...

$\mathbb{R} = \mathbb{1} + \mathbb{N}$

near identity $\mathbb{R} = \mathbb{1} + \delta \mathbb{N} = \mathbb{1} + \eta$

$R^j{}_h R^j{}_i = \delta_{hi}$

$(\delta^j{}_h + \eta^j{}_h)(\delta^j{}_i + \eta^j{}_i) =$

$\delta^j{}_h \delta^j{}_i + \delta^j{}_h \eta^j{}_i + \eta^j{}_h \delta^j{}_i + \eta^2 =$

$\delta^h{}_i + \eta^h{}_i + \eta^i{}_h = \delta^h{}_i$

so

$\eta^h{}_i = -\eta^i{}_h \Rightarrow \eta = -\eta^T \Rightarrow \eta^i{}_i = 0$
(no sum)

Then $$\xi^i \to \xi^{i\prime} = (\delta^i{}_j + \eta^i{}_j)\,\xi^j$$

$$\xi^i + \delta\xi^i = \delta^i{}_j\,\xi^j + \eta^i{}_j\,\xi^j$$

$$\delta\xi^i = \eta^i{}_j\,\xi^j$$

In 3d:

$$\delta\xi^1 = \cancel{\eta^1{}_1}\,\xi^1 + \eta^1{}_2\,\xi^2 + \eta^1{}_3\,\xi^3$$

$$\delta\xi^2 = \eta^2{}_1\,\xi^1 + \cancel{\eta^2{}_2}\,\xi^2 + \eta^2{}_3\,\xi^3$$

$$\delta\xi^3 = \eta^3{}_1\,\xi^1 + \eta^3{}_2\,\xi^2 + \cancel{\eta^3{}_3}\,\xi^3$$

for i, j, k cyclic, can cast in terms of one index

$$\eta^i{}_j = \delta\alpha_k \quad \text{so}$$

$$\alpha = \begin{pmatrix} 0 & \delta\alpha_3 & -\delta\alpha_2 \\ -\delta\alpha_3 & 0 & \delta\alpha_1 \\ \delta\alpha_2 & -\delta\alpha_3 & 0 \end{pmatrix}$$

and

$$\left.\begin{aligned} \delta\xi^1 &= \delta\alpha_3\,\xi^2 - \delta\alpha_2\,\xi^3 \\ \delta\xi^2 &= -\delta\alpha_3\,\xi^1 + \delta\alpha_1\,\xi^3 \\ \delta\xi^3 &= \delta\alpha_2\,\xi^1 - \delta\alpha_3\,\xi^2 \end{aligned}\right\} \quad \delta\xi^i = \xi^j\alpha_k - \xi^k\alpha_j$$

$$i \neq k \neq j$$

Again, $\delta\xi^i = u^i{}_\sigma\, \delta\alpha^\sigma$ can pick 'em off

$$\delta\xi^1 = u^1{}_1\, \delta\alpha^1 + u^1{}_2\, \delta\alpha_2 + u^1{}_3\, \delta\alpha^3$$

so

$$u^1{}_1 = 0$$
$$u^1{}_2 = -\xi^3 = -u^2{}_1$$
$$u^1{}_3 = \xi^2 = -u^3{}_1$$

to make the generators $\quad X_\sigma = u^i{}_\sigma \dfrac{\partial}{\partial\xi^i}$

$$X_1 = u^1{}_1 \frac{\partial}{\partial\xi^1} + u^2{}_1 \frac{\partial}{\partial\xi^2} + u^3{}_1 \frac{\partial}{\partial\xi^3}$$

$$X_1 = \xi^3 \frac{\partial}{\partial\xi^1} - \xi^2 \frac{\partial}{\partial\xi^3} \qquad \text{etc}$$

and can just pound out the algebra

$$[X_1, X_2] = X_3$$

$$[X_2, X_3] = X_1$$

$$[X_3, X_1] = X_2$$

conventionally $\quad J_\sigma \equiv i X_\sigma$

$$[J_\sigma, J_\rho] = i\, \varepsilon_{\sigma\rho\pi} J^\pi$$

This group is $SO(3)$ — "special" det $R = +1$; orthogonal; 3d

det $R = \pm 1$: $O(3)$

$O(3) = SO(3) \otimes J$

$I_s \vec{x} = -\vec{x}$

$\boxed{\text{Example 3.}}$ another transformation on a vector:

$$\xi^i \to \xi^{i'} = \mathcal{U}(\alpha)^i{}_j\, \xi^j$$

ξ - 2d vector. $\mathcal{U}$: unitary, unimodular

$$\mathcal{U}\mathcal{U}^+ = 1$$

$$\sum_j \mathcal{U}^i{}_j \,(\mathcal{U}^{*k}{}_j)^* = \delta^{ik}$$ <u>not</u> necessarily <u>real</u>

$$\downarrow$$

$$\mathcal{U}^1{}_1 \mathcal{U}^1{}_1^* + \mathcal{U}^1{}_2 \mathcal{U}^1{}_2^* = 1$$

$$\mathcal{U}^2{}_1 \mathcal{U}^2{}_1^* + \mathcal{U}^2{}_2 \mathcal{U}^2{}_2^* = 1$$

$$\mathcal{U}^1{}_1 \mathcal{U}^2{}_1^* + \mathcal{U}^1{}_2 \mathcal{U}^2{}_2^* = 0$$

try $\mathcal{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad \mathcal{U}^+ = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$

$$\mathcal{U}\mathcal{U}^+ = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ a^*c + b^*d & cc^* + dd^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$|a|^2 + |b|^2 = 1$

$|c|^2 + |d|^2 = 1$ $\qquad$ unimodular $\Rightarrow$ $ad - bc = 1$

$ac^* + bd^* = 0$

so, generally $\qquad \mathcal{U} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ $\quad \leftarrow$ operates in a vector space of <u>spinors</u>, ξ

Infinitesimal: $\mathbb{U} = \mathbb{1} + \mathfrak{u}$

$= \mathbb{1} + \eta_{\mathbb{1}}$

$$\xi'^i = (\delta^i{}_j + \eta^i{}_j)\xi^j$$
$$= \xi^i + \delta\xi^i$$
$$= \xi^i + \eta^i{}_j \xi^j$$

3 parameters required

$$\eta_{\mathbb{1}} = \begin{pmatrix} -i\alpha_1 & \alpha_2 + i\alpha_3 \\ -\alpha_2 + i\alpha_3 & -i\alpha_1 \end{pmatrix}$$

As before.. $X_\sigma = u^i{}_\sigma \frac{\partial}{\partial \xi^i}$

$$\rightarrow \quad [X_\sigma, X_\rho] = 2i\,\varepsilon_{\sigma\rho\pi} X^\pi$$

define $X_\sigma \equiv -2i S_\sigma$

$$[S_\sigma, S_\rho] = i\,\varepsilon_{\sigma\rho\pi} S^\pi$$

SU(2) 2d

special unitary

<u>same</u> <u>algebra</u> as SO(3) $\rightarrow$ some "ism" going on!

Obviously – spin $S_\sigma = \frac{1}{2}\sigma_\rho$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SU(2)

special unitary 2 dimensions

"Fold the 3-d $\vec{x}$ operated on by SO(3) into 2d imaginary vectors operated on by SU(2)

Form: $\mathbb{H} = \vec{x} \cdot \vec{\sigma} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$

w/ ME's

$$h^i{}_j = (x\cdot\sigma)^i{}_j$$

$$\mathbb{H} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$ ← and $\mathbb{H}$ can stand for $\vec{x}$, element by element.

Now, imagine transforming $\vec{x}$ by actually transforming $\mathbb{H}$

Need $\det \mathbb{H} = -x_3^2 - (x_1 - ix_2)(x_1 + ix_2) = -x_3^2 - x_1^2 - x_2^2 =$
$= -\text{length}(\vec{x})$

$\Rightarrow$ $\mathbb{H}$ can stand for $\vec{x}$!

To transform a matrix.. need an $\mathbb{A}$

$$\mathbb{H}' = \mathbb{A}\,\mathbb{H}\,\mathbb{A}^{-1} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

2d restrict $\mathbb{A} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ like SU(2)

$$\mathbb{H}' = \begin{pmatrix} aa^*x^3 + a^*bx^1 + ia^*b\,x^2 + ab^*x^1 - iab^*x^2 - bb^*x^3 & \\ & \end{pmatrix}$$

so: $x^3 \to x^{3\prime} = h'^1{}_1 = x^1(a^*b + ab^*) + ix^2(a^*b - ab^*)$
$+ x^3(aa^* - bb^*)$

etc.

Since its $\vec{x}$ we could write it as a 3x3

After all, it's $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We <u>could</u> write it as
$\vec{x}' = \mathbb{R}\vec{x}$

$$\mathbb{R} = \begin{pmatrix} a^*b + ab^* & i(a^*b - ab^*) & aa^* - bb^* \\ & & \\ & & \end{pmatrix}$$

turns out it's: real, orthogonal $\det \mathbb{R} = +1$
The properties of a matrix representation of $SO(3)$

Further: had we done $\mathbb{B} = -\mathbb{A}$
$\mathbb{H}' = \mathbb{B}\mathbb{H}\mathbb{B}^{-1}$

$\Rightarrow$ <u>same</u> $\mathbb{R}$

So: are <u>2</u> $SU(2)$ related to <u>1</u> $SO(3)$. <u>Homomorphic</u>

NEXT PAGE

X Infinitesimal transformations - remember

$$f(\vec{\xi}\,') = f(\vec{\xi}) + df(\vec{\xi})$$

$$df(\vec{\xi}) = f(\vec{\xi}\,') - f(\vec{\xi}) = \delta\alpha^\sigma X_\sigma f(\vec{\xi})$$

$$X_\sigma f(\vec{\xi}) = \lim_{\delta\alpha \to 0} \frac{f(\vec{\xi}\,') - f(\xi)}{\delta\alpha^\sigma}$$

What are the vectors?

What about the transformed ξ – the spinor representation

Write as $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ which are complex ← not like x^i.

Can also have $\xi^+ \to \xi^{+\prime} = \xi^+ u^+$

which transform differently

$$\begin{pmatrix} \xi^{1\prime} \\ \xi^{2\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$$

$$\begin{pmatrix} \xi^{1\prime} \\ \xi^{2\prime} \end{pmatrix} = \begin{pmatrix} a\xi^1 + b\xi^2 \\ -b^*\xi^1 + a^*\xi^2 \end{pmatrix}$$

and

$$(\xi^{1*\prime}, \xi^{2*\prime}) = (\xi^{1*}, \xi^{2*}) \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

$$= (\underbrace{\xi^{1*}a^* + \xi^{2*}b^*}_{\xi^{1*\prime}}, \underbrace{-b\xi^{1*} + a\xi^{2*}}_{\xi^{2*\prime}})$$

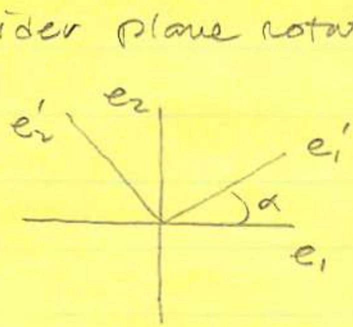
rename $\left.\begin{array}{l} -\xi^{2*} \equiv \eta^1 \\ \xi^{1*} \equiv \eta^2 \end{array}\right\}$ and then

$$\eta^{1\prime} = a\eta^1 + b\eta^2$$
$$\eta^{2\prime} = -b^*\eta^1 + a^*\eta^2$$

$\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}$ transforms like $\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$

important for antiparticle representations BACK

Consider plane rotation, SO(2)



$$|e_i'\rangle = |e_j\rangle R^i{}_j$$

$$\mathbb{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

and in components $\quad x^{i\prime} = R^i{}_j x^j$

Consider small angle $\quad f(\vec{x}\,') = f(x', y')$

$$= f(x\cos\delta\alpha - y\sin\delta\alpha,\ x\sin\delta\alpha + y\cos\delta\alpha)$$

Taylor-expand around identity

$$f(x', y') = f(x, y) - (x - x')\frac{\partial f}{\partial x} - (y - y')\frac{\partial f}{\partial y} + \ldots$$

from $\quad df = \delta x^\sigma X_\sigma f \quad$ in the limit:

$$X f(\vec{x}) = \lim_{\delta\alpha \to 0} \frac{-(x - x')\frac{\partial f}{\partial x} - (y - y')\frac{\partial f}{\partial y}}{\delta\alpha}$$

$$= \lim_{\delta\alpha \to 0} \frac{-(x - \cos\delta\alpha - y\sin\delta\alpha)\frac{\partial f}{\partial x} - (y - \text{etc})\frac{\partial f}{\partial y}}{\delta\alpha}$$

take limit $\quad \cos\delta\alpha \to 1$

$\quad \sin\delta\alpha \to \delta\alpha$

$$X f(\vec{x}\,') = \frac{-y\,\delta\alpha\frac{\partial f}{\partial x} + x\,\delta\alpha\frac{\partial f}{\partial y}}{\delta\alpha} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) f$$

which is the X_3 or J_3 generator again in SO(3)