Lecture 9 What we did

Worked through more examples of how to generate the Lie Algebra for particular groups.

SO(3) rotation group in 3 parameters.
SU(2) special unitary group in 2 parameters.
goes back to the same thing:

$$d\xi^i = u^i_\sigma(\vec{\xi})\,\delta\alpha^\sigma$$

change a parameter... change the vector.

$$X_\sigma \equiv u^i_\sigma \frac{\partial}{\partial \xi^i}$$

reflects how much

generator of the transformation. and satisfies an algebra

$$[X_\rho, X_\eta] = c^\sigma_{\rho\eta} X_\sigma \qquad \text{Lie Algebra.}$$

described $\boxed{\text{THE RECIPE}}$

For SO(3), its: $[X_\sigma, X_\rho] = \varepsilon_{\rho\sigma\pi} X^\pi$

By identifying $J_\sigma \equiv i X_\sigma$, we get the standard prescription for generating rotations.

$$[J_\sigma, J_\rho] = i\,\varepsilon_{\sigma\rho\pi} J^\pi$$

For su(2) ... $[X_\sigma, X_\rho] = 2i\,\varepsilon_{\sigma\rho\pi} X^\pi$

where by defining

$$X_\sigma \equiv -2i\,S_\sigma \qquad S_\sigma = \tfrac{1}{2}\sigma_\sigma$$

$$[S_\sigma, S_\rho] = i\,\varepsilon_{\sigma\rho\pi} S^\pi$$

same algebra as SO(3).

SO(3) induces rotations in 3-space, $\vec{x}$

but

SU(2) can do that as well... 2 ways. 2:1 relationship

SU(2) : SO(3)

homomorphism.

Key to Lie Groups is the idea of an infinitesimal transformation...

$$R_n(\theta) = e^{-i\vec{J}\cdot\vec{\theta}_n} \qquad SO(3) \longrightarrow \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$A_n(\theta) = e^{-i\vec{\sigma}\cdot\vec{\theta}/2} \qquad SU(2) \longrightarrow \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$$

Defined the Rank as the # operators which commute with every generator $\rightarrow$ Casimir Operator., J^2

	SU(2)	SU(3)
rank.	1	2
# commutation relations	1	2
# generators	3	8

Lecture 9.

QM:

The wave function corresponding to a general ψ state vector is

$$\psi(\vec{x}) = \langle \vec{x} | \psi \rangle$$

Rotate the coordinates

$$|\vec{x}\rangle \longrightarrow |\vec{x}'\rangle = \Gamma(R)|\vec{x}\rangle$$

will change the wave function $\psi(\vec{x}) \to \psi'(\vec{x})$

$$|\psi\rangle = \int |\vec{x}\rangle \psi(\vec{x})\, d\vec{x}$$

project $\quad \langle \vec{x}' | \psi \rangle = \int \langle \vec{x}' | \vec{x} \rangle \psi(\vec{x})\, d\vec{x}$

$$= \int \delta(\vec{x} - \vec{x}') \psi(\vec{x})\, d\vec{x}$$
$$= \psi(\vec{x}') = \psi(R^{-1}\vec{x})$$

A Hilbert Space representation —

$$|\psi\rangle \longrightarrow |\psi'\rangle = \Gamma|\psi\rangle$$

If $[H, \Gamma] = 0$ and if $H|\psi\rangle = |\psi\rangle E$

then ψ' are <u>also</u> solutions to

$$[H, \Gamma] = 0 \equiv [H, J_n] = 0 \quad \text{for} \quad \Gamma_n = e^{-i\vec{J}\cdot\hat{n}\theta}$$

This kind of invariance is recognized as angular momentum conservation

The

$|\psi'\rangle$ all transform among themselves by $\Gamma(R)$

$\rightarrow$ a little vector subspace of all vectors. built by angular momentum eigenstates.

label them by a pair of indices which denote & count the basis vector (m) of the n^{th} IRR of SO(3)

$|\psi_m^{(n)}\rangle$

In function space, the rotation operation does

$$|\psi_i^{(n)}\rangle \rightarrow \Gamma(R)|\psi_i^{(n)}\rangle \longrightarrow |\psi_i^{(n)\prime}\rangle = |\psi_j^{(n)}\rangle \underbrace{D^{(n)}(R)^j{}_i}$$

a matrix representation of the group elements of $\mathbb{R}$

For SO(3), $J_r \Rightarrow J_1, J_2 \,\&\, J_3$

and

$$J^2 = J_1^2 + J_2^2 + J_3^2 \longrightarrow \text{the Casimir Operator}$$

From the algebra, $[J_1, J_2] = i\,\varepsilon_{123} J^3 = i J^3$

can show that only 1 can be the diagonalized operator $\rightarrow$ conventional to choose J_3

And,

$$[J^2, J_3] = 0$$

How many simultaneous eigenstates of all of the J operators are there?

Pretend $|\alpha\rangle$ is common to the J_i's

$$J_i|\alpha\rangle = a_i|\alpha\rangle$$

From the Lie Algebra, $[J_1, J_2] = i\,\varepsilon_{123} J^3 = iJ^3$

so, $[J_1, J_2]|\alpha\rangle = (J_1 J_2 - J_2 J_1)|\alpha\rangle$
$= (a_1 a_2 - a_2 a_1)|\alpha\rangle = 0$

$\|$

$iJ_3|\alpha\rangle = i a_3|\alpha\rangle \neq 0$ nope

$\Rightarrow$ there cannot be a state which is simultaneously an eigenstate of all of the J_i.

The custom: choose one, J_3.

and find, by construction, $J^2 = J_1^2 + J_2^2 + J_3^2$

$$[J^2, J_3] = 0$$

so $|\alpha\rangle$ can be an eigenstate of both J^2 and J_3.

More custom... but, you'll see... with a purpose:

Construct the raising and lowering operators.

$$J_\pm = J_1 \pm i J_2 \qquad \begin{cases} [J_3, J_\pm] = \pm J_\pm \\ [J_+, J_-] = 2 J_3 \\ [J^2, J_\pm] = 0 \ \text{(A)} \quad [J^2, J_3] = 0 \ \text{(B)} \end{cases}$$

(A) & (B) $\Rightarrow$ must label states twice $|\xi_m^{(n)}\rangle$

m counts basis states within nth IRR – "weight"

$$J_3 |\xi_m^{(n)}\rangle = m |\xi_m^{(n)}\rangle$$

$$J_3 J_+ |\xi_m^{(n)}\rangle = \{ [J_3, J_+] + J_+ J_3 \} |\xi_m^{(n)}\rangle$$

$$= \{ J_+ + J_+ J_3 \} |\xi_m^{(n)}\rangle$$

$$= (1+m) J_+ |\xi_m^{(n)}\rangle$$

so $J_\pm |\xi_m\rangle$ is a state having J_3 eigenvalue of $(m \pm 1)$

write as

$$J_\pm |\xi_m^{(n)}\rangle = |\xi_{m\pm1}^{(n)}\rangle c_\pm$$

keep operating & eventually run out of states in the subspace. The maximum. $2j+1$

$$|\xi_{m=j}^{(n)}\rangle$$

Usually label the states of IRR by j rather than n

$$|\xi_m^{(j)}\rangle$$

So, by convention, along with the Casimir operator, $|\xi_m^{(j)}\rangle$ is chosen to be an eigenstate — BASIS VECTOR — of J_3

$$J_3 |\xi_m^{(j)}\rangle = |\xi_m^{(j)}\rangle m$$

$$J_\pm |\xi_m^{(j)}\rangle = |\xi_{m\pm1}^{(j)}\rangle c_\pm$$

$$J^2 |\xi_m^{(j)}\rangle = |\xi_m^{(j)}\rangle \lambda$$

From the definitions

$$J^2 = \frac{1}{2}\left(J_+ J_- - J_- J_+ \right) + J_3^2$$

Find λ:

$$\lambda = \langle \xi_m^{(j)} | J^2 | \xi_m^{(j)} \rangle \geq 0$$

$$= \langle \xi_m^{(j)} | J_1^2 + J_2^2 + J_3^2 | \xi \rangle \geq 0$$

$$\lambda = \langle \xi | J_1^2 | \xi \rangle + \langle \xi | J_2^2 | \xi \rangle + m^2 \geq 0$$

So

$$\lambda \geq m^2 \geq 0$$

m bounded by λ

Find $c_\pm$

$$|c_\pm|^2 = \langle \xi^{m(j)} | J_\mp J_\pm | \xi_m^{(j)} \rangle$$

using commutation relations

$$= \langle \xi^{m(j)} | J^2 - J_3^2 + J_3 | \xi_m^{(j)} \rangle$$

$$= \langle \xi^{m(j)} | \lambda - m^2 \mp m | \xi_m^{(j)} \rangle$$

so $\quad c_\pm = \pm\sqrt{\lambda - m^2 \mp m}$

$\pm$ choosen in 1935 by Condon & Shortley.

and we have, top sign.

$$\lambda - m^2 - m \geq 0$$

$$\lambda \geq m(m+1)$$

The maximum value of m can be λ .. choose it to be j. Then

$$\lambda = m_M(m_M+1) \equiv j(j+1)$$

and

$$J^2 |\xi_m^{(j)}\rangle = |\xi_m^{(j)}\rangle j(j+1)$$

and $\quad J_+ |\xi_{m_M}^{(j)}\rangle = |\xi_{m_M+1}^{(j)}\rangle \sqrt{\underbrace{m_M(m_M+1) - m_M^2 - m_M}}$

$$m_M^2 + m_M - m_M^2 - m_M = 0$$

or

$$J_+ |\xi_j^{(j)}\rangle = 0$$

$$J_- |\xi_{-j}^{(j)}\rangle = 0$$

Then, $\quad J^2 |\xi_j^{(j)}\rangle = (J_3^2 + J_3 + \overbrace{J_- J_+}^{=0}) |\xi_j^{(j)}\rangle$

$$= (j^2 + j)|\xi_j^{(j)}\rangle = j(j+1)|\xi_j^{(j)}\rangle$$

& $\quad J^2 |\xi_{-j}^{(j)}\rangle = j(j+1)|\xi_{-j}^{(j)}\rangle \quad$ same

$\Rightarrow$ the value of the Casimir Operator depends only on the IRR's dimension, not the weight.

we've presumed orthonormal states, so can finish $|c_\pm(jm)|^2$

$$= \langle \xi_m^{(j)} | \lambda - m^2 \mp m | \xi_m^{(j)} \rangle$$

$$= \langle \quad | \; j(j+1) - m^2 \mp m \, | \xi_m^{(j)} \rangle$$

$$c_\pm(jm) = \pm\sqrt{j(j+1) - m(m\pm1)}$$

+ by Condon and Shortley convention.

By inspection: $c_+(jj) = c_-(j\,{-j}) = 0$

$-j \leq m \leq j$, differing by an integer

and

$$(m=-j) \leq m \leq (m=j) \Rightarrow 2j+1 \text{ values.}$$

so $j = \dfrac{\text{integer}}{2} = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}$

So, the matrix representations of $SO(3)$ will be $(2j+1)\times(2j+1)$ matrices

and the basis vectors will be $2j+1$ columns or rows

For $|\xi^{(j)}_m\rangle$ a standard notation is $|j\, m\rangle$

eg. For $j = 1/2$ $\quad J_3 |\xi^{(1/2)}_{1/2}\rangle = |\xi^{(1/2)}_{1/2}\rangle \frac{1}{2}$

$$J_3 |\tfrac{1}{2}\, \tfrac{1}{2}\rangle = |\tfrac{1}{2}\tfrac{1}{2}\rangle \tfrac{1}{2}$$

$$J_\pm\, |\tfrac{1}{2}\, \pm\tfrac{1}{2}\rangle = |\tfrac{1}{2}\;\; 1 \pm \tfrac{1}{2}\rangle \sqrt{(1/2 \mp 1/2)(\tfrac{1}{2} \pm \tfrac{1}{2} + 1)}$$

and easily

$$J_+ |\tfrac{1}{2}\, -\tfrac{1}{2}\rangle = |\tfrac{1}{2}\; \tfrac{1}{2}\rangle$$

$$J_- |\tfrac{1}{2}\; 1/2\rangle = |1/2\, -1/2\rangle$$

A convenient spinor basis is $\quad |1/2\; 1/2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\tfrac{1}{2}\, -1/2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with the IRR

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J^2 = 3/4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where $\quad J_i^{(\frac{1}{2})} = \frac{1}{2}\sigma_i$

For $j=1$, do the same thing and find:

$$J_1^{(1)} = \sqrt{1/2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_2^{(1)} = \sqrt{1/2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

$$J_3^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J^{(2)2} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can find other bases.

Remember, from the algebra in coordinate space:

$$L_j = -i\left(x_i \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_i} \right)$$

which when rewritten in spherical coordinates become,

$$L_1 = i\left(\sin\varphi \frac{\partial}{\partial\theta} + \cot\theta \cos\varphi \frac{\partial}{\partial\varphi} \right)$$

$$L_2 = i\left(-\cos\varphi \frac{\partial}{\partial\theta} + \cot\theta \sin\varphi \frac{\partial}{\partial\varphi} \right)$$

$$L_3 = -i \frac{\partial}{\partial\varphi}$$

$$L^2 = -\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right)$$

The basis vectors that span this SO(3) space?

The Y_m^ℓ's. where

$$L_3 Y_\ell^m(\theta, \varphi) = m Y_\ell^m(\theta, \varphi)$$

$$L_\pm Y_\ell^m(\theta, \varphi) = c_\pm(\ell m) Y_\ell^{m\pm 1}(\theta, \varphi)$$

$$L^2 Y_\ell^m(\theta, \varphi) = \ell(\ell+1) Y_\ell^m(\theta, \varphi)$$

Remember, the abstract operation of SO(3):

$$|\xi_m^{(j)}\rangle = R(\alpha\beta\gamma)\,|\xi_m^{(j)}\rangle$$

$$= |\xi_n^{(j)}\rangle D^{(j)}(\alpha\beta\gamma)^n{}_m$$

So, the matrix elements themselves,

$$D^{(j)}(\alpha\beta\gamma)^n{}_m = \langle \xi^{n(j)} |\, R(\alpha\beta\gamma)\, | \xi_m^{(j)} \rangle$$

are called the "Wigner Functions" or "D Functions"

For our finite rotations where J_3 is diagonal,

$$R_i(\theta) = e^{-iJ_i\theta} \quad \text{so}$$

$$R(\alpha\beta\gamma) = e^{-i\alpha J_3}\, e^{-i\beta J_2}\, e^{-i\gamma J_3}$$

where

$$e^{i\gamma J_3}|\xi_m^{(j)}\rangle = \sum_{n=0}^{\infty} \frac{(-i)^n(\gamma)^n}{n!} J_3^n |\xi_m^{(j)}\rangle$$

$$= \sum_{n=1}^{\infty} \frac{(-i)^n(\gamma)^n}{n!} m^n |\xi_m^{(j)}\rangle$$

$$= e^{-i\gamma m}|\xi_m^{(j)}\rangle$$

and the D's can be reduced to

$$D^{(j)}(\alpha\beta\gamma)^n{}_m = e^{-i\alpha n}\langle \xi^{(j)n}| e^{-i\beta J_2} |\xi_m^{(j)}\rangle e^{-i\gamma m}$$

$$\equiv e^{-i\alpha n} \underbrace{d^{(j)}(\beta)^n{}_m}_{\text{"little d's"}} e^{-i\gamma m}$$

These satisfy many orthogonality and symmetry relations related to those we found for the Γ's of the discrete groups

Easy to construct. eg. spin $1/2$

$$d^{(1/2)}(\beta) = e^{-i\frac{\beta}{2}\sigma_2} = \mathbb{1}\cos\beta/2 - i\sigma_2 \sin\beta/2$$

operator

$$= \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

and the actual matrix element.

eg.

$$d^{(1/2)}(\beta)^{1/2}{}_{1/2} = (1,0)\begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \cos\beta/2$$

$$d^{(1/2)}(\beta)^{1/2}{}_{-1/2} = (1\ 0)\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

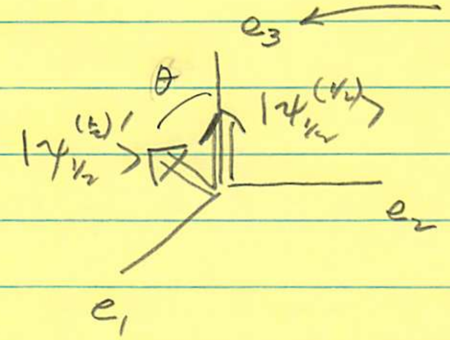
$$= -\sin\beta/2$$

etc.

So,

$$D^{(1/2)}(\alpha\beta\gamma) = \begin{pmatrix} e^{-i\alpha/2}\cos\beta/2\, e^{-i\gamma/2} & -e^{-i\alpha/2}\sin\beta/2\, e^{i\gamma/2} \\ e^{i\alpha/2}\sin\beta/2\, e^{-i\gamma/2} & e^{i\alpha/2}\cos\beta/2\, e^{i\gamma/2} \end{pmatrix}$$

These are actually useful for rotating spinors.



$$R(0\,\theta\,0)\,|{1/2}\,{1/2}\rangle = |\psi^{(1/2)}_n\rangle\, D^{(1/2)}(0\,\theta\,0)^n{}_{1/2}$$

$$= |\psi^{(1/2)}_{-1/2}\rangle\, D^{(1/2)}(0\,\theta\,0)^{1/2}{}_{-1/2}$$

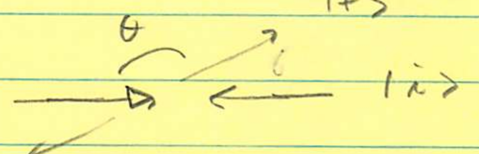
$$+ |\psi^{(1/2)}_{1/2}\rangle\, D^{(1/2)}(0\,\theta\,0)^{1/2}{}_{1/2}$$

$$= |{1/2}\,{-1/2}\rangle \sin\theta/2 + |{1/2}\,{1/2}\rangle \cos\frac{\theta}{2}$$

$$= \begin{pmatrix}0\\1\end{pmatrix}\sin\theta/2 + \begin{pmatrix}1\\0\end{pmatrix}\cos\theta/2$$

$$= \begin{pmatrix}\cos\theta/2\\ \sin\theta/2\end{pmatrix}$$

useful. Sometimes you want to calculate helicity amplitudes



rotate $\langle i|$ to get $\langle f|$

$A \sim \langle f|\,\text{physics}\,|i\rangle$

For $j=1$, the rotation matrices are a chore to calculate. They are

$$d^{(1)}(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{\sqrt{2}} \end{pmatrix}$$

The D's are <u>all</u> matrix representations of $SU(2)$

The $D^{(j)}$'s for $j=1$ are matrix representations of $SO(3)$

again, the 2:1 relationship.

Taken together: The Rotation Group