34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

172a

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
⋮ ⋮			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	J=1, M=+1		
+1/2	+1/2	1		
		J=1, M=0	**J=0, M=0**	
+1/2	−1/2	1/2	1/2	
−1/2	+1/2	1/2	−1/2	
		J=1, M=−1		
−1/2	−1/2	1		

1×1/2

m_1	m_2		
		3/2, +3/2	
+1	+1/2	1	
		3/2, +1/2	**1/2, +1/2**
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
		3/2, −1/2	**1/2, −1/2**
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
		3/2, −3/2	
−1	−1/2	1	

2×1

m_1	m_2			
		3, +3		
+2	+1	1		
		3, +2	**2, +2**	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
		3, +1	**2, +1**	**1, +1**
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
		3, 0	**2, 0**	**1, 0**
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
		3, −1	**2, −1**	**1, −1**
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
		3, −2	**2, −2**	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
		3, −3		
−2	−1	1		

1×1

m_1	m_2			
		2, +2		
+1	+1	1		
		2, +1	**1, +1**	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
		2, 0	**1, 0**	**0, 0**
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
		2, −1	**1, −1**	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
		2, −2		
−1	−1	1		

2×1/2

m_1	m_2		
		5/2, +5/2	
+2	+1/2	1	
		5/2, +3/2	**3/2, +3/2**
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
		5/2, +1/2	**3/2, +1/2**
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
		5/2, −1/2	**3/2, −1/2**
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
		5/2, −3/2	**3/2, −3/2**
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
		5/2, −5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
		2, +2	
+3/2	+1/2	1	
		2, +1	**1, +1**
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
		2, 0	**1, 0**
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
		2, −1	**1, −1**
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
		2, −2	
−3/2	−1/2	1	

3/2×1

m_1	m_2			
		5/2, +5/2		
+3/2	+1	1		
		5/2, +3/2	**3/2, +3/2**	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
		5/2, +1/2	**3/2, +1/2**	**1/2, +1/2**
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
		5/2, −1/2	**3/2, −1/2**	**1/2, −1/2**
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
		5/2, −3/2	**3/2, −3/2**	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
		5/2, −5/2		
−3/2	−1	1		

3/2×3/2

m_1	m_2				
		3, +3			
+3/2	+3/2	1			
		3, +2	**2, +2**		
+3/2	+1/2	1/2	1/2		
+1/2	+3/2	1/2	−1/2		
		3, +1	**2, +1**	**1, +1**	
+3/2	−1/2	1/5	1/2	3/10	
+1/2	+1/2	3/5	0	−2/5	
−1/2	+3/2	1/5	−1/2	3/10	
		3, 0	**2, 0**	**1, 0**	**0, 0**
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4
		3, −1	**2, −1**	**1, −1**	
+1/2	−3/2	1/5	1/2	3/10	
−1/2	−1/2	3/5	0	−2/5	
−3/2	+1/2	1/5	−1/2	3/10	
		3, −2	**2, −2**		
−1/2	−3/2	1/2	1/2		
−3/2	−1/2	1/2	−1/2		
		3, −3			
−3/2	−3/2	1			

2×3/2

m_1	m_2				
		7/2, +7/2			
+2	+3/2	1			
		7/2, +5/2	**5/2, +5/2**		
+2	+1/2	3/7	4/7		
+1	+3/2	4/7	−3/7		
		7/2, +3/2	**5/2, +3/2**	**3/2, +3/2**	
+2	−1/2	1/7	16/35	2/5	
+1	+1/2	4/7	1/35	−2/5	
0	+3/2	2/7	−18/35	1/5	
		7/2, +1/2	**5/2, +1/2**	**3/2, +1/2**	**1/2, +1/2**
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10
		7/2, −1/2	**5/2, −1/2**	**3/2, −1/2**	**1/2, −1/2**
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5
		7/2, −3/2	**5/2, −3/2**	**3/2, −3/2**	
0	−3/2	2/7	18/35	1/5	
−1	−1/2	4/7	−1/35	−2/5	
−2	+1/2	1/7	−16/35	2/5	
		7/2, −5/2	**5/2, −5/2**		
−1	−3/2	4/7	3/7		
−2	−1/2	3/7	−4/7		
		7/2, −7/2			
−2	−3/2	1			

2×2

m_1	m_2					
		4, +4				
+2	+2	1				
		4, +3	**3, +3**			
+2	+1	1/2	1/2			
+1	+2	1/2	−1/2			
		4, +2	**3, +2**	**2, +2**		
+2	0	3/14	1/2	2/7		
+1	+1	4/7	0	−3/7		
0	+2	3/14	−1/2	2/7		
		4, +1	**3, +1**	**2, +1**	**1, +1**	
+2	−1	1/14	3/10	3/7	1/5	
+1	0	3/7	1/5	−1/14	−3/10	
0	+1	3/7	−1/5	−1/14	3/10	
−1	+2	1/14	−3/10	3/7	−1/5	
		4, 0	**3, 0**	**2, 0**	**1, 0**	**0, 0**
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5
		4, −1	**3, −1**	**2, −1**	**1, −1**	
+1	−2	1/14	3/10	3/7	1/5	
0	−1	3/7	1/5	−1/14	−3/10	
−1	0	3/7	−1/5	−1/14	3/10	
−2	+1	1/14	−3/10	3/7	−1/5	
		4, −2	**3, −2**	**2, −2**		
0	−2	3/14	1/2	2/7		
−1	−1	4/7	0	−3/7		
−2	0	3/14	−1/2	2/7		
		4, −3	**3, −3**			
−1	−2	1/2	1/2			
−2	−1	1/2	−1/2			
		4, −4				
−2	−2	1				

d functions

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

so, for example

$$|1\,0\rangle = \sqrt{1/2}\;|1/2\;\, -1/2\rangle + \sqrt{1/2}\;|-1/2\;\, 1/2\rangle$$

It works backwards

$$|1/2\;\, -1/2\rangle = \sqrt{1/2}\;|1\,0\rangle + \sqrt{1/2}\;|0\,0\rangle$$

I said there were 3 ways:

② explicitly use the lowering operators.

start with the highest weight state, $j_1 = j_2 = 1/2$ again

$$|11\rangle = |1/2\ 1/2\rangle_1\, |1/2\ 1/2\rangle_2 \qquad \text{C.G.C.} \equiv 1$$

$$J_-|11\rangle = \left(J_-(1)\,|1/2\ 1/2\rangle\right)|1/2\ 1/2\rangle_2 + |1/2\ 1/2\rangle_1 \left(J_-(2)\,|1/2\ 1/2\rangle_2\right)$$

$$= \underbrace{\sqrt{(1/2 + 1/2)(1/2 - 1/2 + 1)}}_{=1} \left\{ |1/2\ {-1/2}\rangle_1\, |1/2\ 1/2\rangle_2 + |1/2\ 1/2\rangle_1\, |1/2\ {-1/2}\rangle_2 \right.$$

$$\sqrt{(1+1)(1-1+1)}\,|10\rangle = \sqrt{2}\,|10\rangle$$

so, $$|10\rangle = \sqrt{1/2}\,|1/2\ {-1/2}\rangle_1\, |1/2\ 1/2\rangle_2 + \sqrt{1/2}\,|1/2\ 1/2\rangle_1\, |1/2\ {-1/2}\rangle_2$$

easy. quick. cumbersome.

Tensorial Approach

-general-

labels the weight.

$$\xi^i \rightarrow \xi'^i = U^i{}_j \xi^j$$

write: $\xi^1 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^2 \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the <u>fundamental</u> of $SU(2)$

Can build product states to higher-ranked tensors

$$\psi^{ij} = \xi^i \otimes \eta^j \qquad \text{or just} \qquad \psi^{ij} = \xi^i \eta^j$$

generally reducible w/ C.G. coefficients.

$$\psi^{ij} \longrightarrow \psi'^{ij} = U^i{}_k U^j{}_l \xi^k \eta^l = U^i{}_k U^j{}_l \psi^{kl}$$

Raising & lowering operators are non-tensorial in nature — they change the weight values

$$(\text{lower})\, \xi^i = (\text{lower})\, \xi^1 = \xi^2$$
$$(\text{lower}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

represented by

$$\underbrace{e(ij)}_{\text{just a name}}{}^{a}{}_{\underbrace{b}_{\text{matrix elements}}} \equiv \delta_i{}^a \delta^j{}_b$$

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \end{matrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \end{matrix}$$

a

so

$$\xi^{1,a} \longrightarrow \begin{matrix} \xi^{1,1} = 1 \\ \xi^{1,2} = 0 \end{matrix} \qquad \xi^{2,b} \longrightarrow \begin{matrix} \xi^{2,1} = 0 \\ \xi^{2,2} = 1 \end{matrix}$$

so

$$e(ij)^a{}_b \, \xi^{k,b} = \begin{pmatrix} \delta_i{}^1 \delta^j{}_1 & \delta_i{}^1 \delta^j{}_2 \\ \delta_i{}^2 \delta^j{}_1 & \delta_i{}^2 \delta^j{}_2 \end{pmatrix} \begin{pmatrix} \xi^{k,1} \\ \xi^{k,2} \end{pmatrix}$$

so, for:

$$e(12)^a{}_b \, \xi^2 = \begin{pmatrix} \delta_1{}^1 \delta^2{}_1 \xi^{2,1} + \delta_1{}^1 \delta^2{}_2 \xi^{2,2} \\ \delta_1{}^2 \delta^2{}_1 \xi^{2,1} + \delta_1{}^2 \delta^2{}_2 \xi^{2,2} \end{pmatrix}$$

$$= \begin{pmatrix} \xi^{2,2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \xi^1$$

compact form:

$$e(ij)\, \xi^k = \xi^i \delta_j{}^k$$

<u>not</u> a tensor equation
<u>not</u> specific to $SU(2)$

$$e(12)\, \xi^2 = \xi^1 \delta_2{}^2 = \xi^1$$
$$e(12)\, \xi^1 = \xi^1 \delta_2{}^1 = 0$$
$$e(21)\, \xi^1 = \xi^2 \delta_1{}^1 = \xi^2$$

For product states: $\psi^{ij} = \xi^i \eta^j$

The highest weight state is $\psi^{11} = \xi^1\eta^1$ which is irreducible.

Get the next weight by lowering

$$e(21)\psi^{11} = [e(21)\xi^1]\eta^1 + \xi^1[e(21)\eta^1]$$

$$= \xi^2\delta_1{}^1\eta^1 + \xi^1\eta^2\delta_1{}^1$$

$$= \xi^2\eta^1 + \xi^1\eta^2$$

$$= \psi^{21} + \psi^{12} \equiv \psi^{(12)}$$ symmetric combination

again:

$$e(21)\psi^{(12)} = [e(21)\xi^2]\eta^1 + \xi^2[e(21)\eta^1]$$

$$+ [e(21)\xi^1]\eta^2 + \xi^1[e(21)\eta^2]$$

$$= \cancelto{0}{\xi^2\delta_1{}^2\eta^1} + \xi^2\eta^2\delta_1{}^1 + \xi^2\delta_1{}^1\eta^2 + \cancelto{0}{\xi^1\eta^2\delta_1{}^2}$$

$$= \xi^2\eta^2 + \xi^2\eta^2 = 2\xi^2\eta^2$$

So, $\psi^{(11)} = \xi^1\eta^1$
$\psi^{(12)} = \xi^2\eta^1 + \xi^1\eta^2$
$\psi^{(22)} = 2\xi^2\eta^2$

normalize ⟶ $\xi^1, \eta^1 \to \xi^1, \eta^1$
$\xi^2, \eta^2 \to \sqrt{1/2}\,\xi^1, \sqrt{1/2}\,\eta^1$

& set

$$\psi^{(11)} = \xi^1\eta^1$$
$$\psi^{(12)} = \sqrt{1/2}\,(\xi^2\eta^1 + \xi^1\eta^2)$$
$$\psi^{(22)} = \xi^2\eta^2$$

CG coefficients: normalization constants. Familiar?

If dimensionality of representation is d (= # weights)

$$\#\text{ symmetric combinations} = \tfrac{1}{2}d(d+1)$$

For 2, 2d states, the total number is 4. We have 3 and the 4th must be constructed to be orthogonal.

We can always do this by the symmetry of the indices:

$$\psi^{ij} = \tfrac{1}{2}(\xi^i\eta^j + \xi^j\eta^i) + \tfrac{1}{2}(\xi^i\eta^j - \xi^j\eta^i) = \xi^i\eta^j$$
$$= \tfrac{1}{2}\psi^{(ij)} + \tfrac{1}{2}\psi^{[ij]} \leftarrow \text{antisymmetric}$$

The total # antisymmetric states:

$$\# A = d^2 - \tfrac{1}{2}d(d+1) = \tfrac{1}{2}d(d-1) \to 1 \text{ here.}$$

$$\psi^{[12]} = \sqrt{1/2}\,(\xi^1\eta^2 - \xi^2\eta^1)$$