# Worksheet #9 - PHY102 (Spr. 2008)

**Due Friday March 21st 6pm**

## More on "Do loops", Illustrating Chaos

**Tools that you need**

**Do** (you can also use Table or NestList)

You will also need to learn how to plot lists of numbers using:

**ListPlot**

**The new physics - Chaos**

Chaos, though discussed extensively for a couple of centuries (e.g. Boltzmann and Maxwell discussed "molecular chaos"), has really come into its own since the widespread use of computers. An early surprise is that even quite simple looking systems can have chaos, whereas it was originally thought that chaos only occured in systems with billions of molecules. In this worksheet you will study perhaps the simplest system which shows chaos, namely the "mapping"

$$x_{n+1} = \lambda x_n (1 - x_n) \qquad (1)$$

This mapping models, for example, how a population density, $x_{n+1}$, changes as a function of the number of generations, $n$. Actually, it is not a very realistic model but it does illustrate many of the features of more complex systems. The parameter $\lambda$ can be considered to be the "birth rate", ie. the number of offspring from the last generation. Anyway the way it works is that if we know the population density at some time and call that density $x_0$, then the population density of the next generation is $x_1 = \lambda x_0(1 - x_0)$. This procedure is continued using Eq. 1 to find the population density for later generations. Intuitively, chaos means lack of order. Mathematically, it is defined by how stable the behavior of a set of equations is to small perturbations in the initial conditions. In the context of equation 1 this means, how stable are the set of iterates $(x_0, x_1, x_2, x_3 ...)$ when you make the small change $x_0 -> x_0^\delta = x_0 + \delta x_0$. If this change is made, we get a new set of iterates $(x_0^\delta, x_1^\delta, x_2^\delta, x_3^\delta ...)$. If a set of equations is in a chaotic regime then the divergence of trajectories is exponential with a positive "Lyapunov" exponent. In the context of our example,

$$|x_n^\delta - x_n| \approx e^{\nu n}, \qquad (2)$$

where, in a *truly chaotic* system, the Lyapunov exponent $\nu$ is positive. In that case small changes in the initial conditions lead to very large changes in the final behavior. This is the "butterfly effect" which makes it very difficult to compute the precise dynamical behavior of complex dynamical systems, such as the weather or the stock market.

**Problem**

(i) Write a Mathematica code to iterate the mapping (Eq. 1). Plot the behavior of $x_n$ as a function $n$ for $\lambda = 2.5, \lambda = 3.5, \lambda = 3.95$. Discuss the behavior you observe.

(ii) By extracting the steady state behavior at 100 values in the range $1 < \lambda < 4$, plot the steady state behavior as a function of $\lambda$ on one graph. This is achieved by plotting $x_n$ vs $\lambda$ at these 100 values.

(iii) Consider to close starting values of $x_1$, namely $x_1 = 0.51$ and $x_1 = 0.510000000000001$. Using these starting values find the way in which the trajectories diverge as $n$ increases. From this data estimate the Lyapunov exponent using Eq. 2.