

Lecture 16: Centripetal Acceleration, $a_c = v^2/r$

Circular motion at constant speed, v leads to $a_c = v^2/r$

It may surprise you to learn that when a mass moves around a circle of radius r at constant speed, v , it is still accelerating because its DIRECTION is changing. This change in direction leads to a change in velocity because velocity is a vector! When velocity changes, there is always an acceleration given by,

$$\vec{a}_{av} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t} \quad (1)$$

By using similar triangles it is possible to show that (see page 199 of text and/or go to lecture),

$$\frac{\Delta v}{v} = \frac{\Delta s}{r} \quad (2)$$

so that

$$\vec{a}_c = \frac{v \Delta s}{r \Delta t} = \frac{v^2}{r} \quad (3)$$

where a_c is the magnitude of the centripetal acceleration. Its direction is toward the center of the circle. This acceleration must be provided by a force, which in the case of a mass on a string is provided by the tension in the string. If this force is removed, the mass goes in a straight line. In the case of a car going around a corner, the force is produced by the tires on the road.

Some notes on directions

Note that centripetal acceleration a_c is NOT the same as angular acceleration α . The angular acceleration leads to a linear acceleration $a = \alpha r$, which is TANGENTIAL to the circle. Centripetal acceleration is toward the CENTER of the circle, which is often called the radial direction. The angular velocity and angular acceleration have a direction which is positive counterclockwise, which you can remember by using the right hand rule. Circular motion counterclockwise in the plane has a direction which is along the positive z axis.

The gravitational force

It is worth stating again that there are only four fundamental forces. The strong force and the weak force operate at length scales of order the

size of a nucleus (1fermi), the electromagnetic force and the gravitational force dominate at larger length scales than that. We have learned about forces like friction and the force on a spring. Both of those forces come from electromagnetic and gravitational forces, if we look closely enough. The magnitude of the gravitational force between two spherical masses m_1 and m_2 is, to a very good approximation, given by,

$$F = \frac{Gm_1m_2}{r^2} \quad (4)$$

where $G = 6.673 \times 10^{-11} Nm^2/kg^2$ and r is the distance between the centers of the spheres. The gravitational force is always attractive (there is no antigravity!). In a famous experiment Cavandish (1798) measured G by studying the force between two masses as a function of their separation. The gravitational acceleration of mass m_2 due to mass m_1 is,

$$a_1 = F/m_2 = \frac{Gm_1}{r^2} \quad (5)$$

If we calculate this value for $m_1 = \text{mass of earth}$ and $r = \text{radius of earth}$, then we find $g = 9.8m/s^2$. Note also that Newton's third law applies here, so the force on mass m_1 due to mass m_2 is equal and opposite to the force on m_2 due to m_1 .

Gravitational potential energy

We have so far said that the change in gravitational potential energy near the surface of earth is $\Delta PE = mgh$. This is a valid equation when we consider a small mass near the surface of a large mass. However there is a much more general formula, given by,

$$PE = -\frac{Gm_1m_2}{r} \quad (6)$$

The potential energy gets smaller (more negative) as two masses get closer together. When two masses are a large distance apart, the potential energy goes to zero. This must reduce to mgh near the surface of a large spherical mass and it is shown in the textbook, page 212 that it does.

Motion in a gravitational field

Now that we know the gravitational force and the gravitational potential, we can apply Newton's second law or the work-energy theory to solve a lot of problems.

Circular Orbits

Consider motion of a planet (e.g. earth) around the sun. Take the mass of the planet to be m and the mass of the sun to be M . If the planet is at a distance r from the sun, then the force on the planet due to the sun is the gravitational force. Furthermore, we can consider that the sun is stationary, as its mass is so large compared to that of the planet that its motion is scarcely altered due to the force of the planet on the sun. Now let's apply Newton's second law,

$$\frac{GMm}{r^2} = ma \quad (7)$$

What acceleration is this?? If the planet is moving at constant angular velocity, ie $\alpha = 0$, then the only acceleration that the earth experiences is the centripetal acceleration. We can then find the velocity using the centripetal acceleration formula,

$$\frac{GMm}{r^2} = m \frac{v^2}{r}. \quad (8)$$

This is a very important relation because it gives a relation between the radius of the orbit and the speed at which a planet moves. As we shall see later, it also demonstrated Kepler's laws, which were known experimentally at the time.

Escape velocity

How fast do we need to fire a projectile (mass m), from the earth surface $R_e = 6,400\text{km}$, $M_e = 6 * 10^{24}\text{kg}$. in order for it to escape the earth's gravitational field? We can solve this using the work energy theory, $\Delta PE + \Delta KE = 0$. The projectile is finally at rest, so $KE_f = 0$. Once the projectile has "escaped" its potential energy is $PE_f = 0$, we then have,

$$-PE_i + KE_i = 0 \quad (9)$$

Using the gravitational potential energy equation, we find that,

$$\frac{GM_e m}{R_e} = \frac{1}{2} m v_{esc}^2 \quad (10)$$

Note that this assumes that there is no viscous drag on the projectile, which in this case is a very poor approximation. However we could do a similar calculation for MARS and it would be OK. Anyway assuming there is no drag, we get,

$$v_{esc} = \left(\frac{2GM_e}{R_e}\right)^{1/2} = 25,000mph \quad (11)$$

This is a lower limit on the velocity required. We can also estimate the amount of energy which this requires, ie $0.5mv^2$. If you do the calculation it is clear that it is a large number. For this reason high energy density rocket fuel is used.