PHY481 - Lecture 13

A line charge near a grounded conducting cylinder

A line charge λ at position x_0 on the x-axis is near a grounded conducting cylinder, of radius R, which has its center at $(x, y) = (0, 0)$, with its central axis along the z-axis. The line charge lies parallel to the central axis of the cylinder. Find the electrostatic potential for $r > R$ in plane cylindrical co-ordinates r, ϕ. Note that the potential does not depend on z.

Using Gauss's law it is easy to show that the electric field near a uniform line charge is $\vec{E}(r) = \lambda\hat{r}/(2\pi\epsilon_0 r)$. The potential is then of the form $V(r) = \lambda ln(constant/r)/(2\pi\epsilon_0)$. The constant is chosen to fit the boundary conditions.

Let's assume that the problem of a line charge near a grounded conducting cylinder is solved by using an image charge which is located at position x_0' and with charge per unit length $-\lambda$. Now we need to find x_0' and we need to check that $V(r, \phi) = 0$, and that $\vec{E}(R, \phi) = E_r\hat{n}$. The potential is given by superposition, so that,

$$V(r, \phi) = \frac{\lambda}{2\pi\epsilon_0}[ln(c_1/r_1) - ln(c_1/r_2)] + c_2 \tag{1}$$

where c_1 and c_2 are constants. Using the cosine rule we have,

$$r_1^2 = r^2 + x_0^2 - 2rx_0cos\phi; \quad r_2^2 = r^2 + {x_0'}^2 - 2rx_0'cos\phi \tag{2}$$

Lets also assume that the reciprocal relation holds (why not!!), ie $x_0' = R^2/x_0$. We then have

$$V(r, \phi) = \frac{\lambda}{4\pi\epsilon_0}ln(\frac{r^2 + R^4/x_0^2 - 2r(R^2/x_0)cos\phi}{r^2 + x_0^2 - 2rx_0cos\phi}) + c_2 \tag{3}$$

At the surface of the cylinder we have,

$$V(R, \phi) = \frac{\lambda}{4\pi\epsilon_0}ln([\frac{R^2}{x_0^2}](\frac{R^2 + x_0^2 - 2Rx_0cos\phi}{R^2 + x_0^2 - 2Rx_0cos\phi})) + c_2 \tag{4}$$

This must be zero for our solution to be correct, which implies that,

$$c_2 = -\frac{\lambda}{2\pi\epsilon_0}ln(R/x_0) \tag{5}$$

The solution to our problem is then,

$$V(r, \phi) = \frac{\lambda}{4\pi\epsilon_0}ln(\frac{r^2 + R^4/x_0^2 - 2rR^2/x_0cos\phi}{r^2 + x_0^2 - 2rx_0cos\phi}) - \frac{\lambda}{2\pi\epsilon_0}ln(R/x_0) \tag{6}$$

or

$$V(r,\phi) = \frac{\lambda}{4\pi\epsilon_0} ln(\frac{x_0^2 r^2 + R^4 - 2rx_0R^2cos\phi}{r^2 + x_0^2 - 2rx_0cos\phi}) - \frac{\lambda}{2\pi\epsilon_0} ln(R) \tag{7}$$

Now we need to check that the electric field is given correctly. We find that,

$$E_\phi = \frac{-1}{r}\frac{\partial V}{\partial \phi} = \frac{-\lambda}{4\pi\epsilon_0 r}(\frac{2rx_0R^2sin\phi}{x_0^2r^2 + R^4 - 2rx_0R^2cos\phi} - \frac{2rx_0sin\phi}{r^2 + x_0^2 - 2rx_0cos\phi}) \tag{8}$$

From Eq. (7) it is evident that $V(R,\phi) = 0$ as required and from Eq. (8) we find $E_\phi(R,\phi) = 0$. We have therefore found a solution which satisfies the boundary condition, so it is correct. For completeness, the electric field in the radial direction is given by,

$$E_r = -\frac{\partial V}{\partial r} = \frac{-\lambda}{4\pi\epsilon_0}(\frac{2rx_0^2 - 2x_0R^2cos\phi}{x_0^2r^2 + R^4 - 2rx_0R^2cos\phi} - \frac{2r - 2x_0cos\phi}{r^2 + x_0^2 - 2rx_0cos\phi}) \tag{9}$$

Closing remarks on generalizing image charge problems

We have solved three basic image charge problems: (i) a point charge near a grounded flat conducting surface (Lecture 12) (ii) a point charge near a grounded conducting sphere (Lecture 12) (iii) a line charge near a grounded conducting cylinder (This Lecture). Lets call these solutions $V_G(\vec{r})$.

The extension to problems where the conductor is at some finite voltage (instead of zero) requires adding charges to produce that voltage. The charges have to be placed symmetrically to ensure that no electric field is generated in the metal. E.g. if we want a sphere of radius R at potential V_0, then we place an image charge Q_0 at the center of the sphere so that $V_0 = kQ_0/R$. This corresponds to distributing the charge Q_0 uniformly on the surface of the sphere. The electrostatic potential for $r > R$ of this problem is found by superposition, i.e. $V(\vec{r}) = V_G(\vec{r}) + kQ_0/r$. In the case of a conducting slab, a sheet of image charge is placed at the center of the slab, while in the case of a conducting cylinder, a line charge is placed at the center of the cylinder.

In a similar way, if we are given a problem where a point charge is near an isolated conducting sphere, cylinder or slab which has total charge Q, then we again have to place an image charge at the center of the sphere. However, now the magnitude of the image charge at the center of the metal sphere has to be the sum of the total charge on the sphere plus the value of the image charge of the grounded system. For example an isolated conducting sphere of charge Q requires that an image charge of $Q - q'$ be placed at its center, so the total potential for $r > R$ becomes $V_G(\vec{r}) + k(Q - q')/r$, where $q' = -qR/z_0$ is the image charge of the grounded sphere.

Finally there are problems where we are asked to consider a point charge (spherical cavity) or line charge (cylindrical cavity) inside a cavity that is totally surrounded by metal. Again the metal can be grounded, or at a fixed potential V_0 or have a total charge Q. The basic solution is for the grounded case where the only induced charge is on the inner surface of the metal. The other cases are treated using superposition as before. For example, this leads to an interesting effect for an isolated sphere, which has a spherical cavity. No matter where we place a point charge inside the sphere, the induced charge on the outer surface $(-q')$ is symmetric!! The induced charge on the inner surface of the metal is not symmetric - it has a non-trivial $\sigma(\theta)$ in general. However in order to ensure that $\vec{E} = 0$ in the metal, the charge on the outer surface must be placed uniformly on the surface of the sphere.

The solution to the grounded case of a point charge inside a spherical cavity inside a metal is the same as that of the point outside of a metal sphere, however we have to be careful in changing the variables correctly. The real charge in the case of the spherical cavity corresponds to the image charge in the case of the metal sphere. Similarly for the case of the conducting cylinder and the cylindrical cavity. Note also that if there is no charge inside a cavity inside a metal, then no charge is induced on the surfaces of the cavity, no matter how many charges are placed near the the exterior surfaces of the metal.