PHY481 - Lecture 22

Chapter 8 of PS, Chapter 5 of Griffiths

A. Basics of Magnetostatics

Magnetostatics is the study of static magnetic fields and the steady currents (DC) that generate them. The MKS unit of magnetic field is the Tesla (T) and often the magnetic field is given in Gauss (G), the CGS unit. The relation is $10,000G = 1T$, so small fields are often given in Gauss.

In the early 1800's several researchers noted that there is a force between two parallel wires that carry a steady current . If the wires are separated by distance d,

$$\frac{|F|}{l} = \frac{\mu_0}{2\pi d} i_1 i_2 \quad \text{attractive for currents in the same direction} \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7} N/A^2$ is the permeability of vacuum. If the currents are in opposite directions the wires repel. This result can be compared with the force between two infinite parallel line charges, with charge densities λ_1, λ_2. The magnitude of the force between the line charges, when separated by d, is given by,

$$\frac{|F|}{l} = \frac{q_2}{l}|E_{\lambda_1}| = \frac{\lambda_1 \lambda_2}{2\pi\epsilon_0 d} \quad \text{repulsive for like charges} \tag{2}$$

where we used the electric field near a line charge $E(r) = \lambda/2\pi\epsilon_0 r$. Other than the direction of the force these two results are similar. Clearly, excess charge leads to electric fields and a DC leads to magnetic fields. The total force is a superposition of these two effects, so that electrostatics and magnetostatics don't affect each other.

Ampere and Oersted also noticed that magnets are affected by a DC current in a wire and Ampere found that,

$$\vec{B}(r) = \frac{\mu_0 i}{2\pi r}\hat{\phi} \quad \text{Magnetic field near a wire} \tag{3}$$

This is not such a leap as we already know that the electric field due to a line charge is

$$\vec{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r} \quad \text{Electric field near a line charge} \tag{4}$$

The most remarkable difference between these two results is the *direction* of the fields

- The electric field diverges from the line charge and is curl free ($\vec{\nabla} \wedge \vec{E} = 0$)

- The magnetic field forms circles around the steady current and is divergence free ($\vec{\nabla} \cdot \vec{B} = 0$) Notice that the magnetic field is not moving even though the steady current is flowing and even though we often talk about the circulation of the magnetic field about the current-The magnetic field is static. The direction of circulation is given by the right hand rule, applied to either the magnetic field or to the current.

B. Ampere's law and related problems

We start with Section 4 of PS or Section 5.3.3 of Griffiths by writing down Ampere's law. Ampere realised that his measurements for the magnetic field near a wire may be written in the form of a path integral.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i = \mu_0 \int \vec{j} \cdot d\vec{A} \quad Ampere's\ law \tag{5}$$

Here $d\vec{l}$ is a small vector along the direction of the path. For example for a circle the unit vector is tangent to the circle. Now consider the magnetic field around a long straight wire, where the current, i, is in the $\hat{k}$ direction, then we have,

$$\int_0^{2\pi} B(r)\hat{\phi} \cdot r d\phi \hat{\phi} = 2\pi r B(r) = \mu_0 i \tag{6}$$

Solving for $B(r)$ yields,

$$\vec{B}(r) = \frac{\mu_0 i}{2\pi r}\hat{\phi} \tag{7}$$

Ampere's law is correct only for DC currents. Later in the course we will add an additional term which enables us to use this equation even when there are time dependent fields - the Maxwell displacement current term.

There is a set of important problems that can be solved analytically using the integral form of Ampere's law. The most basic is the line charge case as illustrated above. Another simple case is a sheet of current. In that case, consider a sheet of current lying in the x-y place, with the current directed along the positive x-direction, so that the surface current density is $K\hat{i}$, where K is the current per unit length. If we draw an $L \times H$ rectangular amperian loop around the current sheet, with L the length of a section parallel to the y-axis and H a section parallel to the z-direction.

$$2LB = \mu_0 LK; \quad \text{so that} \quad \vec{B} = -\frac{1}{2}\mu_0 K\hat{j},\ z > 0; \quad \vec{B} = \frac{1}{2}\mu_0 K\hat{j},\ z < 0. \tag{8}$$

A practically important and closely related case is that of an infinite solenoid of radius R with its axis along the z-axis, where there are n "turns" per unit length. In that case, the field outside the solenoid is, by symmetry, zero. The finite field inside the solenoid is found from,

$$LB = \mu_0 Lni; \quad \text{so that} \quad B = \mu_0 ni \quad r < 0 \tag{9}$$

where i is the current in the circuit. Using the RHR, if the solenoid current is rotating in the $\hat{\phi}$ direction, the magnetic field is along the $\hat{k}$ direction. Large magnetic fields can be achieved by increasing n, or by increasing i, which is the basic goal in design of electromagnets. Copper electromagnets can achieve magnetic fields of up to $10T$, though values around $1 - 5T$ are more standard. Superconducting electromagnets can achieve higher values, but require liquid Helium cooling. Large magnets used in physics experiments, such as the NSCL, LHC etc, use superconducting electromagnets. Use of magnets in NMR and in medicine (MRI) also require large magnetic fields and rely on electromagnets. Electromagnets provide very nice control of the size of the magnetic field. William Stugeon invented the electromagnet in 1823. High termperature superconductors hold the promise of superconducting magnets operating at liquid Nitrogen instead of Liquid He which would be a very important advance.

Another class of problems that can be solved with Ampere's law is the case of a wire with current density $j(r)$ up to its radius R, with one example being a constant current density j_0 for $r < R$. It is a useful excersize to do this case and find the magnetic field in the two regimes; $B(r) = \mu_0 j_0 r/2$ for $r < R$; and $B(r) = \mu_0 j_0 R^2/2r$ for $r > R$.

C. Biot-Savart law and related problems

Ampere's law is convenient for cases with high symmetry, but we need a more general law for cases where the current carrying wire is not so symmetric, for example in current loops. The Bio-Savart law is the generalization of Ampere's law to these cases and is given by,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{l} \wedge \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{id\vec{l} \wedge \vec{r}}{r^3}; \quad \text{Biot} - \text{Savart law} \tag{10}$$

Notice that this looks a lot like Coulomb's law,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dQ\hat{r}}{r^2}. \tag{11}$$

The complication is that in the Biot-Savart case we now have to do the cross-product, but in practice that is not so hard. Integration of the Biot-Savart law is used with to find the magnetic field due to wires carrying direct currents.

Lets illustrate this with an example that we already solved using Ampere's law, an infinite straight wire. Now consider the magnetic field around a long straight wire, where the current i is in the $\hat{k}$ direction. We use cylindrical co-ordinates and we want to find the magnetic field as a function of r. We have (for $z > 0$),

$$d\vec{B}(r,z) = \frac{\mu_0 i}{4\pi}\frac{dz\vec{k}\wedge\hat{R}}{R^2} = \frac{\mu_0 i}{4\pi}\frac{dzsin(180-\alpha)}{R^2}\hat{\phi} \tag{12}$$

where α is the angle between $\hat{k}$ and $\vec{R}$ and $R^2 = r^2 + z^2$. A similar expression applies to $z < 0$, but with $sin(180-\alpha) \to sin(\alpha)$. Since $sin(\alpha) = sin(180-\alpha)$ the contributions for $z < 0$ and $z > 0$ add. We also have $sin(\alpha) = r/R$. The magnetic field at position r is found by integrating $d\vec{B}$, so that,

$$\vec{B}(r) = \frac{\mu_0 i}{4\pi}\int_{-\infty}^{\infty}\frac{r\ dz}{(r^2+z^2)^{3/2}}\hat{\phi} \tag{13}$$

The integral,

$$\int_{-\infty}^{\infty}\frac{r\ dz}{(r^2+z^2)^{3/2}} = \frac{2}{r} \quad \text{implies that} \quad \vec{B}(r) = \frac{\mu_0 i}{2\pi r}\hat{\phi} \tag{14}$$

which of courses agrees with the result of using Ampere's law.

Other cases where the Biot-Savart law is integrated to find the magnetic field include, circular and rectangular loops, discs and spheres. A common problem is to take a disc or sphere with a constant charge density and to spin it at a constant rate. This leads to currents that generate a static magnetic field.