PHY481 - Lecture 8

Sections 3.5-3.7 of PS

A. The electric potential - continued

Mathematical Derivation

The electrostatic potential can also be deduced on purely mathematical grounds using the relation $\vec{\nabla} \wedge \vec{E} = 0$. This equation is satisfied when $\vec{E} = -\vec{\nabla} V$ due to the vector identity $\vec{\nabla} \wedge (\vec{\nabla} V) = 0$ which holds for any scalar function V. In terms of the electrostatic potential, the differential form of Gauss's law then becomes,

$$\nabla^2 V = -\rho/\epsilon_0. \tag{1}$$

This is Poisson's equation and is the most commonly solved form of Gauss's law. The special case $\rho = 0$ is also very important and is called Laplace's equation. In fact the whole of Chapter 5 of PS is devoted to Laplace's equation. Laplace's equation is most non-trivial when boundary conditions corresponding to finite charge or electostatic potential are used. We then need to solve boundary value problems of partial differential equations.

Methods for calculating the electrostatic potential

1. Direct method: Find the electric field by using e.g. Gauss's law and then carry out the integral,

$$V(r) = -\int_\infty^r \vec{E} \cdot d\vec{l} \tag{2}$$

If this integral diverges, we cannot use a reference potential at infinity, but instead can only talk of voltages across finite distances. For example if the electric field is a constant and directed in the $\hat{r}$ direction, then we have,

$$V_{ab} = -\int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} \to -E(r_b - r_a) = -Ed \quad \text{if E is constant} \tag{3}$$

where $d = r_b - r_a$.

2. Superposition: Do a scalar addition of the contributions from each charge,

$$V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|} \to \int d\vec{r}' \frac{k\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \tag{4}$$

where the last expression applies to a continuous charge distribution with charge density ρ.

3. Differential form: Here we use Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{5}$$

To complete our mathematical skills lets introduce the delta function, which allows us to treat discrete charge distributions using Poisson's equation. The delta function has the properties,

$$\delta(\vec{r} - \vec{r_i}) = 1 \quad \text{if} \quad \vec{r} = \vec{r_i}. \quad \text{It is zero otherwise} \tag{6}$$

In addition the integral of the delta function is one

$$\int \delta(\vec{r} - \vec{r_i}) d\vec{r} = 1 \tag{7}$$

For a point charge at position $\vec{r_i}$, we then write,

$$\rho(\vec{r}) = q_i \delta(\vec{r} - \vec{r_i}) \tag{8}$$

Since the potential for a point charge at position $\vec{r_i}$ is $kQ/|\vec{r} - \vec{r_i}|$, then we already know the solution to the differential equation,

$$\nabla^2 G(\vec{r} - \vec{r'}) = -\delta(\vec{r} - \vec{r'}) \tag{9}$$

It is $G(\vec{r} - \vec{r'}) = 1/(4\pi|\vec{r} - \vec{r'}|)$ where G is the Green's function. That is, $q_i * G/\epsilon_0$ is the potential for a point charge q_i. It is then clear that for a general charge density we have,

$$V(\vec{r}) = \frac{1}{\epsilon_0} \int d\vec{r'} \rho(\vec{r'}) G(|\vec{r} - \vec{r'}|) \tag{10}$$

These manipulations seem pretty redundant in this problem, however the concept of a Green's function is central to higher level developments in all branches of physics. In electrostatics, the Green's function is the solution to the smallest element of charge and superposition then gives the general solution for any charge distribution.

There are many ways to construct a function which obeys these relations and this is great for analytic work. In fact a lot of theoretical physics requires working with delta functions. Problem 3.19 of the third homework uses one form of a one dimensional delta function. Another one which is used a lot is the Gaussian. Note that the delta function is defined by a limiting process in both cases. There are other forms which use integrals to achieve the same effect. Choosing the right form of the delta function in complex problems is quite an art that takes some effort to master.

(i) Electric potential due to a point charge

The electric field due to a point charge is

$$\vec{E} = \frac{kQ\hat{r}}{r^2} \tag{11}$$

The electric potential due to a point charge is then,

$$V = -\int_{\infty}^{r} \frac{kQ\hat{r}}{r'^2} \cdot \vec{ds} = -\int_{\infty}^{r} \frac{kQ\hat{r}}{r'^2} \cdot dr'\hat{r} = -kQ\int_{\infty}^{r} \frac{dr'}{r'^2} = \frac{kQ}{r} \tag{12}$$

(ii) Electric potential due to an electrostatic dipole

The electric potential and electric field of an electrostatic dipole are given by,

$$V(\vec{r}) = k\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{and} \quad \vec{E}(\vec{r}) = k\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} \quad r >> d \tag{13}$$

In problem 3.37 you will calculate the electric field from the potential. Actually you already did this in Assignment 1, so it is just a reminder. Below we explicitly show how the expression for the dipole potential is calculated and why it is only valid for $r >> d$. You can solve problem 3.38 of the homework using a similar expansion, except in that case the expansion has to be taken to order $(d/r)^2$.

Consider a dipole lying on the z-axis, with the positive charge at $z = d/2$ and the negative charge at $z = -d/2$. The charges have magnitude q. A point in the x-y plane is defined by its radial distance to the origin, r, and by an angle θ between the direction $\hat{r}$ and the z-axis. We want to find the electric potential due to the dipole as a function, $V(r,\theta)$. The exact expression for the potential of a dipole is then,

$$V(r,\theta) = kq(\frac{1}{r_+} - \frac{1}{r_-}) \tag{14}$$

where,

$$r_-^2 = (\frac{d}{2})^2 + r^2 - 2\frac{d}{2}rcos(180-\theta) = (\frac{d}{2})^2 + r^2 + drcos(\theta) \tag{15}$$

and

$$r_+^2 = (\frac{d}{2})^2 + r^2 - 2\frac{d}{2}rcos(\theta) = (\frac{d}{2})^2 + r^2 - drcos(\theta) \tag{16}$$

Using these expressions, we rewrite Eq. (8) as,

$$V(r,\theta) = \frac{kq}{((\frac{d}{2})^2 + r^2)^{1/2}}[(\frac{1}{1 - \frac{dr}{(\frac{d}{2})^2 + r^2}cos\theta})^{1/2} - (\frac{1}{1 + \frac{dr}{(\frac{d}{2})^2 + r^2}cos\theta})^{1/2}] \tag{17}$$

At long distances where $d/r \to 1$, we can use the expansions,

$$\frac{1}{(1-\delta)^{1/2}} \approx 1 + \frac{\delta}{2} \quad \text{and} \quad \frac{1}{(1+\delta)^{1/2}} \approx 1 - \frac{\delta}{2} \tag{18}$$

where

$$\delta = \frac{drcos\theta}{(\frac{d}{2})^2 + r^2)} \to \frac{d}{r}cos\theta \tag{19}$$

This yields,

$$V(r >> d, \theta) = \frac{kq}{r}[1 + \frac{dcos(\theta)}{2r} - (1 - \frac{dcos(\theta)}{2r})] \tag{20}$$

$$= \frac{kqdcos(\theta)}{r^2} = k\frac{\vec{p} \cdot \hat{r}}{r^2} \tag{21}$$

where $p = qd$ is the magnitude of the dipole moment.

Now lets calculate the electric field, which we have done before using Cartesion co-ordinates. Now we do the calculation using polar co-ordinates. Take the z-axis along the axis of the dipole, then we have,

$$V(\vec{r}) = k\frac{pcos\theta}{r^2}; \quad \text{and} \quad \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \tag{22}$$

Using polar co-ordinates, we have,

$$\vec{E}(\vec{r}) = -(\hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{rsin\theta}\frac{\partial V}{\partial \phi} = \hat{r}\frac{2pkcos\theta}{r^3} + \hat{\theta}\frac{kpsin\theta}{r^3} \tag{23}$$

Now we note that $\hat{p} = \hat{r}cos\theta - \hat{\theta}sin\theta$ so that,

$$\vec{E}(\vec{r}) = \hat{r}\frac{3pkcos\theta}{r^3} - \hat{p}\frac{kp}{r^3} = \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{4\pi\epsilon_0 r^3} \tag{24}$$

Notice that it is a lot easier to do this calculation in polar co-ordinates.

Excersizes: Calculate the potential energy as a function of distance for a uniform sheet of charge, a uniform cylindrical shell of charge, a uniform spherical shell of charge. This is achieved by doing the integral of the electric field. Repeat these calculations for a uniform slab of charge, a uniform cylinder of charge and a uniform sphere of charge