Outline of solutions to Homework 2

Problem 2.6: By symmetry, $\vec{E} = (0, 0, E_z)$, and by superposition we have,

$$E_z = \int \frac{kdQ(\phi)}{r^2} cos(\theta) = \int_0^R \int_0^{2\pi} k \frac{\sigma s d\phi ds}{r^2} cos(\theta) \quad (1)$$

where $r^2 = s^2 + z^2$ and $cos(\theta) = z/r$. We then have,

$$E_z \int_0^R \int_0^{2\pi} kz \frac{\sigma s d\phi ds}{(s^2 + z^2)^{3/2}} = 2\pi kz\sigma \frac{-1}{(s^2 + z^2)^{1/2}} |_0^R = 2\pi kz\sigma [\frac{1}{z} - \frac{1}{(z^2 + R^2)^{1/2}}] \quad (2)$$

As $z \to \infty$, to leading order this is zero. To find the behavior at the next order, we expand $1/(z(1 + (R/z)^2)^{1/2} \approx (1/z)(1 - 0.5(R/z)^2)$, so that $E_z \approx \pi kz\sigma R^2/z^3 = kQ/z^2$, where $Q = \pi R^2\sigma$ is the total charge on the disc, as expected. We can also take the limit $R >> z$, and in that case the electric field reduces to $\sigma/(2\epsilon_0)$ the result for an infinite sheet of charge.

Problem 2.11: The charge on the uniform shell is $Q = 4\pi R^2\sigma$. Using Gauss's law with a spherical Gaussian surface of radius r, we have, $\vec{E}(\vec{r}) = E(r)\hat{r}$, with

$$E(r)4\pi r^2 = q_{encl}/\epsilon_0 \quad (3)$$

For Gaussian surfaces having radius $r < R$, $q_{encl} = 0$, so $E(r < R) = 0$. For Gaussian surfaces with $r > R$, $q_{encl} = Q$, so $E(r) = Q/(4\pi\epsilon_0 r^2)$. That is, the electric field inside a uniform shell of charge is zero and the field outside the shell is like that of a point charge at the origin. These are the so called "shell theorems". It is more work to prove them using superposition.

Problem 2.13: Choose a cylindrical Gaussian surface centered on the wire. The electric field is directed radially in cylindrical co-ordinates, so by symmetry, $\vec{E} = (E(s), 0, 0) = E(s)\hat{s}$. There is no flux through the top or bottom of the cylindrical Gaussian surfaces. The only flux is through the rounded surface, so that,

$$E(s)2\pi sz = \frac{\lambda z}{\epsilon_0}, \quad \text{so that } \vec{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s} \quad (4)$$

Problem 2.18: First find the electric field inside a sphere of radius R, with uniform charge density ρ. As usual for spherical symmetry $\phi_E(r) = 4\pi r^2 E(r)$. The charge enclosed by a sphere of radius r is $\rho Volume = \rho 4\pi r^3/3$. The electric field inside the sphere, i.e. for $r < R$ is $\vec{E} = \rho r/(3\epsilon_0)\hat{r}$. Now assume that the sphere with positive charge is centered at the origin, while the sphere with negative charge is centered at position $\vec{d}$. The electric field due to the negatively charge sphere is then $\vec{E}' = -\rho r'/(3\epsilon_0)\hat{r}'$. This may be written in terms of $\vec{r}$ and $\vec{d}$ by noting that $\vec{r} - \vec{r}' = \vec{d}$, so that using superposition we find,

$$\vec{E}_{total} = \vec{E} + \vec{E}' = \frac{\rho\vec{r}}{3\epsilon_0} - \frac{\rho(\vec{r} - \vec{d})}{3\epsilon_0} = \frac{\rho\vec{d}}{3\epsilon_0} \quad (5)$$

Problem 2.21: From problem 2.18, we have the electric field inside a uniform sphere of charge density ρ as $\vec{E} = \rho\vec{r}/(3\epsilon_0)$. The electric field outside the sphere is just that of a point charge, i.e. $\vec{E} = kQ\vec{r}/r^3$, where $Q = 4\pi R^3\rho/3$. The potential is found by integration,

$$V(r) = -\int_\infty^r \vec{E} \cdot d\vec{l} = -\int_\infty^r E(r')dr'\hat{r} \cdot \hat{r} = -\int_\infty^r E(r')dr' \quad (6)$$

We then have,

$$V(r > R) = -\int_\infty^r \frac{kQ}{r'^2} dr' = \frac{kQ}{r'}|_\infty^r = \frac{kQ}{r}. \quad (7)$$

Inside the sphere we have,

$$V(r < R) = \frac{kQ}{R} - \int_R^r dr' = \frac{kQ}{R} - \int_R^r \frac{\rho r'}{3\epsilon_0} dr' = \frac{kQ}{R} + \frac{\rho}{6\epsilon_0}(R^2 - r^2) \quad (8)$$

Using $Q = 4\pi\rho R^3/3$, this reduces to

$$V(r < R) = \frac{kQ}{R} + \frac{kQ}{2R^3}(R^2 - r^2) = \frac{kQ}{2R}(3 - \frac{r^2}{R^2}) \tag{9}$$

Problem 2.22: From class, we have the electric field outside, $\vec{E} = E(s)\hat{s}$, with $E(s) = \lambda/(2\pi\epsilon_0 s)$, so the potential is given by,

$$V(s,a) = -\int_a^s \frac{\lambda}{2\pi\epsilon_0 s} ds = \frac{\lambda}{2\pi\epsilon_0} ln(a/s) \tag{10}$$

Problem 2.28: We must find the potential at any point inside a uniformly charged sphere using superposition. Without loss of generality, we can choose $\vec{r}$ to lie on the z-axis. The superposition expression for the potential is then,

$$V(r) = \int k\frac{dQ(\theta',\phi')d\vec{r}'}{|\vec{r}-\vec{r}'|} = k\rho\int_0^R\int_0^\pi\int_0^{2\pi}\frac{r'^2 sin\theta' d\theta' d\phi'}{(r'^2 + r^2 - 2r'rcos(\theta'))^{1/2}} \tag{11}$$

The ϕ' integral gives 2π, while the θ' integral is also easy as $2Rr\sin(\theta')$ is the derivative of the argument of the denomenator, so we find,

$$V(r) = 2\pi k\rho\int_0^R dr'\frac{r'^2}{r'r}(r'^2 + r^2 - 2r'rcos(\theta'))^{1/2}|_0^\pi, \tag{12}$$

which reduces to,

$$V(r) = 2\pi k\rho\int_0^R dr'\frac{r'}{r}[(r'^2 + r^2 + 2r'r)^{1/2} - (r'^2 + r^2 - 2r'r)^{1/2}] = 2\pi k\rho\int_0^R dr'\frac{r'}{r}[r' + r - |r' - r|] \tag{13}$$

For $r > r'$, we take the negative sign in the modulus, while for $r < r'$, we take the positive sign in the modulus. If $r > R$, we are always in the regime where $r > r'$, so we take the positive sign. Doing the integral then yields $V(r > R) = kQ/r$, where we used $Q = 4\pi R^3\rho/3$. For $r < R$, we have to break the integral up into two parts so that,

$$V(r) = 2\pi k\rho[\int_0^r dr' 2\frac{r'^2}{r} + \int_r^R dr' 2r'] = 4\pi k\rho[\frac{r^3}{3r} + \frac{1}{2}(R^2 - r^2)] = \frac{kQ}{2R}(3 - \frac{r^2}{R^2}) \tag{14}$$

Problem 2.29: We need to take the Laplacian of $1/|\vec{r} - \vec{r}'|$. However, we know that if $V(r) = kQ/|\vec{r} - \vec{r}'|$, then we must have $\nabla^2 V = -Q\delta(\vec{r} - \vec{r}')/\epsilon_0$, Therefore $\nabla^2(1/|\vec{r} - \vec{r}'|) = 4\pi\delta^3(\vec{r} - \vec{r}')$. Therefore

$$\nabla^2[k\int\frac{\rho(\vec{r}')d\vec{r}'}{|\vec{r}-\vec{r}')}] = k\int\rho(\vec{r}')d\vec{r}'\nabla^2[\frac{1}{|\vec{r}-\vec{r}'|}] = -k\int\rho(\vec{r}')d\vec{r}'(4\pi\delta(\vec{r}-\vec{r}')) = -4\pi k\rho(\vec{r}) = -\rho(\vec{r})/\epsilon_0 \tag{15}$$

Problem 2.32: a) Using Eq. (2.43) of Griffiths, we have,

$$U = \frac{1}{2}\int_0^R 4\pi r^2 dr[\rho\frac{kQ}{2R}(3 - \frac{r^2}{R^2})] = \frac{\rho Q R^2}{5\epsilon_0} \tag{16}$$

b) Using Eq. (2.43) of Griffiths, we use the electric fields $E(r < R) = \rho\vec{r}/(3\epsilon_0)$; $E(r > R) = kQ/r^2$, so that,

$$U = \frac{\epsilon_0}{2}[\int_0^R 4\pi r^2 dr\frac{\rho^2}{9\epsilon_0^2}r^2 + \int_R^\infty 4\pi r^2 dr\frac{Q^2}{16\pi^2\epsilon_0^2 r^4}] = \frac{\epsilon_0}{2}[\frac{4\pi\rho^2}{9\epsilon_0^2}\frac{R^5}{5} + \frac{Q^2}{4\pi\epsilon_0^2}\frac{1}{R}] = \frac{\rho Q R^2}{5\epsilon_0} \tag{17}$$

c) Equation (44) may be evaluated for any Gaussian surface, so lets choose a convenient one, $r = R$. Then the volume integral is just the first part of that in b). The surface integral is $kQ/R \times kQ/R^2 \times 4\pi R^2$, so we find,

$$U = \frac{\epsilon_0}{2}[\frac{\rho Q}{15\epsilon_0^2}R^2 + \frac{4\pi k^2 Q^2}{R}] = \frac{\rho Q R^2}{5\epsilon_0} = \frac{3kQ^2}{5R} \tag{18}$$

In the limit $a \to \infty$, Eq. (2.44) is the same as Eq. (2.45) of Griffiths. Lets show this for the uniform sphere case by considering a spherical volume of radius $a > R$. Now we have to split the volume integral into two parts, so that,

$$\int_\nu E^2 d\tau = 4\pi[\int_0^R r^2 dr\frac{\rho^2 r^2}{9\epsilon_0^2}dr + \int_R^a r^2 dr(\frac{kQ}{r^2})^2] = 4\pi[\frac{\rho^2 R^5}{45\epsilon_0^2} + \frac{(kQ)^2}{R} - \frac{(kQ)^2}{a}] \tag{19}$$

In the limit $a \to \infty$ this reduces to Eq. (17). The surface integral is

$$\oint V\vec{E} \cdot d\vec{a} = 4\pi a^2 (\frac{kQ}{a})(\frac{kQ}{a^2}) = 4\pi \frac{(kQ)^2}{a} \tag{20}$$

In the limit $a \to \infty$, the surface integral goes to zero, so only the volume integral is left and we arrive at Eq. (2.45).

Problem 2.36: a) Draw a spherical Gaussian surface around either one of the cavities, centered at the origin of that cavity, and with the surface passing in the metal region between the cavities. The electric field is zero everywhere on the Gaussian surface, because the Gaussian surface lies within the metal regions. Then the net charge must be zero within the Gaussian surface and hence the charge at the cavity surface must be $-q_a$ for the cavity containing charge q_a and $-q_b$ for the cavity containing q_b. Moreover the charge is uniformly distributed on the surfaces of the cavities. By charge neutrality, a charge of $q_a + q_b$ must be distributed on the outer surface of the metal object, moreover the charge must be uniformly distributed on this surface to ensure that the electric field generated by this outer shell of charge produces zero electric field within the metal. b) The field outside the conductor is $\vec{E} = k(q_a + q_b)\hat{r}/r^2$. c) The field within each cavity is that of a point charge $\vec{E}_a = kq_a(\vec{r} - \vec{r}_a)/|\vec{r} - \vec{r}_a|^3$ and $\vec{E}_b = kq_b(\vec{r} - \vec{r}_b)/|\vec{r} - \vec{r}_b|^3$. d) There is no force on either charge q_a or on charge q_b as within either cavity, there is no net electric field due to any of the shells of charge. e) If a charge q_c were brought near the conductor, the charge distribution at the outer surface would be altered so as to nullify the electric field due to q_c. The precise charge distribution would depend on the location of q_c. Note that the total charge at the outer surface would still be $q_a + q_b$, though there might be regions of the surface that could have a negative charge density.

Problem 2.38: Here we use $\vec{F} = q\vec{E}$, and find the total force on the upper hemisphere by integration. First we note that by symmetry the force is in the $z-$direction. We also have to find the electric field at the surface. As noted in lecture this is one half the electric field just outside the surface. The surface has charge density σ and total charge $Q = 4\pi R^2\sigma$. The force on the upper hemisphere is,

$$F_z = \int dQ(\theta', \phi')E_z(\theta', \phi') = \int_0^{\pi/2} \int_0^{2\pi} \sigma R^2 sin(\theta')d\theta'\phi' \frac{1}{2} \frac{kQcos(\theta')}{R^2}$$

$$= \frac{2\pi R^2 kQ\sigma}{4} \int_0^{\pi/2} sin(2\theta')d\theta' = \frac{2\pi R^2 kQ\sigma}{4} = \frac{kQ^2}{8R^2} \tag{21}$$

Problem 2.39: Capacitance of a coaxial. Use $Q = CV$, where Q is the charge on the coaxial and V is the voltage difference. The voltage difference is given by $\lambda ln(b/a)/(2\pi\epsilon_0)$, so the capacitance is

$$C = \frac{\lambda L}{\lambda ln(b/a)/(2\pi\epsilon_0)} \tag{22}$$

so the capacitance per unit length $C/L = 2\pi\epsilon_0/ln(b/a)$.