PHY481 - Outline of solutions to Homework 3

Problem 3.8: We consider a charge outside a conducting sphere that is neutral. In order that the sphere be neutral, we have to introduce a new image charge $Q_0 = -q'$. The solution for this case is a superposition of the case of a grounded conductor, as solved in Lecture 11, plus the effect of the potential and force produced by Q_0. By placing the second fictitious or image charge, Q_0, at the origin of the conducting sphere we produce an additional potential $V_0 = kQ_0/R$ at the surface of the sphere and kQ_0/r for $r > R$. The conductor is still an equipotential as required and the electric field remains normal to the surface of the conductor, so it is the correct solution. In this case the force on the charge q placed outside the neutral conductor is,

$$\vec{F}_q = [\frac{kqq'}{(a-b)^2} - \frac{kqq'}{a^2}]\hat{r} = [\frac{-kq^2(R/a)}{(a-R^2/a)^2} + \frac{kq^2(R/a)}{a^2}]\hat{r} = kq^2[\frac{R}{a^3} - \frac{Ra}{(a^2-R^2)^2}]\hat{r} \tag{1}$$

The question asks for the force of attraction which is $-\vec{F}_q$.

Problem 3.9: The potential of a line charge is $-\lambda ln(s)/(2\pi\epsilon_0) + constant$. Using Cartesian co-ordinates in two dimensions (in the y-z plane, as the line charge lies along the x-axis) and assuming that the image charge is $-\lambda$ at $z = -d$, we find the solution to be,

$$V(y,z) = \frac{-\lambda}{4\pi\epsilon_0}[ln(y^2 + (z-d)^2) - ln(y^2 + (z+d)^2)] + constant \tag{2}$$

The boundary condition $V(z = 0) = 0$ is satisfied if $constant = 0$. It is also easy to show that $E_y(z = 0) = 0$ so that the parallel electric field at the surface is zero. The electric field is then normal to the surface and the normal component $E_n = E_z$, where,

$$E_z = \frac{\lambda}{4\pi\epsilon_0}[\frac{2(z-d)}{y^2+(z-d)^2} - \frac{2(z+d)}{y^2+(z+d)^2}] \tag{3}$$

The induced surface charge is then,

$$\sigma = \epsilon_0 E_z(z=0) = \frac{\lambda}{\pi}\frac{-d}{y^2+d^2} \tag{4}$$

The total induced charge (per unit length) is then,

$$\lambda_{induced} = \int_{-\infty}^{\infty} \frac{\lambda}{\pi}\frac{-ddy}{y^2+d^2} = \frac{-\lambda d}{\pi}\frac{1}{d}tan^{-1}(y/d)|_{-\infty}^{\infty} = -\lambda \tag{5}$$

Problem 3.10 This problem is solved by using three image charges: $-q$ at $\vec{r}_1 = (-a, b)$; q at $\vec{r}_2 = (-a, -b)$; $-q$ at $\vec{r}_3 = (a, -b)$. The potential in the positive quadrant is then,

$$V(x,y) = \frac{kq}{((x-a)^2+(y-b)^2+z^2)^{1/2}} + \frac{kq}{((x+a)^2+(y+b)^2+z^2)^{1/2}}$$

$$-\frac{kq}{((x-a)^2+(y+b)^2+z^2)^{1/2}} - \frac{kq}{((x+a)^2+(y-b)^2+z^2)^{1/2}} \tag{6}$$

The force on the real charge due to the image charges is,

$$\vec{F}_q = -\frac{kq^2(\vec{r}-\vec{r}_1)}{(\vec{r}-\vec{r}_1)^3} + \frac{kq^2(\vec{r}-\vec{r}_2)}{(\vec{r}-\vec{r}_2)^3} - \frac{kq^2(\vec{r}-\vec{r}_3)}{(\vec{r}-\vec{r}_3)^3} \tag{7}$$

By symmetry there is no force in the z-direction. The forces in the x and y directions are,

$$F_q^x = \frac{-kq^2}{4a^2} + \frac{kq^2a}{4(a^2+b^2)^{3/2}}; \quad F_q^y = \frac{-kq^2}{4b^2} + \frac{kq^2b}{4(a^2+b^2)^{3/2}}; \tag{8}$$

The total energy stored in the configuration of charges, including the image charges is,

$$Energy = Work = \frac{1}{4}[-\frac{2kq^2}{2a} - \frac{2kq^2}{2b} + \frac{2kq^2}{2(a^2+b^2)^{1/2}}] \tag{9}$$

The term in square brackets would be the energy stored if it were a configuration of real charges, while the factor of four reduction arises in our problem as electric fields only occur in one quadrant of space.

Problem 3.20: As seen in Lecture 12, the potential outside an uncharged metal sphere in a uniform field is given by,

$$V(r,\theta) = -E_0 cos\theta[r - \frac{R^3}{r^2}] \quad \text{uncharged case} \tag{10}$$

By superposition, the charged case would have an additional term kQ/r. However this assumes that the potential at infinity is zero, while the solution above assumes that the potential at the origin (z=0) is zero. A general solution to allow us to fix the potential at any location is,

$$V(r,\theta) = -E_0 cos\theta[r - \frac{R^3}{r^2}] + \frac{kQ}{r} + C \quad \text{charged case} \tag{11}$$

Now we can choose the potential to be zero anywhere we want. Lets choose the surface of the metal to be at zero potential. Choosing the constant $C = -kQ/R$ ensures that this is the case. Alternatively, we could set $C = 0$ in which case $V = 0$ as $r \to \infty$ and $z = 0$. Either choice is OK as the choice of C does not affect the electric fields or any other physical observables.

Problem 3.22: (complete the Fourier analysis) We use spherical polar co-ordinates where the general solution is,

$$V(r,\theta) = \sum_{l=0}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta) \tag{12}$$

Inside the sphere, we set $B_l = 0$, while outside the sphere we set $A_l = 0$. The potential must be continuous across the sphere surface, so that,

$$A_l R^l = B_l/R^{l+1}, \quad \text{or} \quad B_l = R^{2l+1}A_l \tag{13}$$

We are told that the charge density is σ_0 on the upper hemisphere and $-\sigma_0$ on the lower hemisphere. The charge density is related to the electric field through

$$E_r^{above}(r=R) - E_r^{below}(r=R) = \frac{\sigma}{\epsilon_0} = \sum_l [-lA_l R^{l-1} - \frac{(l+1)B_l}{R^{l+2}}]P_l(cos\theta) = -\sum_l (2l+1)A_l R^{l-1}P_l(cos\theta) \tag{14}$$

Since the argument of the Legendre Polynomial is $u = cos\theta$, all of the polynomials are even in θ and hence satisfy the symmetry of σ about the origin. We are left with the task of finding A_l from the equation,

$$\frac{\sigma}{\epsilon_0} = -\sum_l (2l+1)A_l R^{l-1}P_l(u) \tag{15}$$

To do this, we carry out the following steps: multiply both sides by $P_m(u)$ and then integrate both sides with respect to u over the interval $[-1,1]$. For the LHS of Eq. (16) we get,

$$\int_{-1}^{1} duP_m(u)\frac{\sigma}{\epsilon_0} = \int_{-1}^{0} duP_m(u)\frac{-\sigma_0}{\epsilon_0} + \int_{0}^{1} duP_m(u)\frac{\sigma_0}{\epsilon_0} \tag{16}$$

The interval $[-1,0]$ corresponds to $[\pi,\pi/2]$ in θ, while $[0,1]$ corresponds to $[\pi/2,0]$. The right hand side of Eq. (16) reduces to,

$$-\sum_l (2l+1)\int_{-1}^{1} duA_l R^{l-1}P_m(u)P_l(u) = -2A_m R^{m-1} \tag{17}$$

where we used the othogonality relation

$$\int_{-1}^{1} P_l(u)P_m(u)du = \frac{2}{2l+1}\delta_{ml} \tag{18}$$

with δ_{ml} the Kronecker delta function. We now have an explicit expression for the series coefficients A_l as,

$$A_l = \frac{-\sigma_0}{2\epsilon_0 R^{m-1}}[-\int_{-1}^{0} P_l(u)du + \int_0^1 P_l(u)du] \tag{19}$$

The Legendre polynomials up to order six are: $P_0(u) = 1$; $P_1(u) = u$; $P_2(u) = (3u^2 - 1)/2$; $P_3(u) = (5u^3 - 3u)/2$; $P_4(u) = (35x^4 - 30x^2 + 3)/8$; $P_5(u) = (63u^5 - 70u^3 + 15u)/8$; $P_6(u) = (231u^6 - 315u^4 + 105u^2 - 5)/16$. On the interval $[-1, 1]$, σ is an odd function in u, so all of the even Legendre polynomials (which are even on this interval) multiplied by σ integrate to zero on this interval. The odd Legendre polynomials however integrate to something finite. We thus only need to do three integrals,

$$2\sigma_0 \int_0^1 udu = \sigma_0; \quad 2\sigma_0 \int_0^1 \frac{1}{2}(5u^3 - 3u)du = \frac{-\sigma_0}{4}; \quad 2\sigma_0 \int_0^1 \frac{1}{8}(63u^5 - 70u^3 + 15u)du = \frac{\sigma_0}{8} \tag{20}$$

We then find,

$$A_0 = 0; A_1 = \frac{-\sigma_0}{2\epsilon_0}; A_2 = 0; A_3 = \frac{\sigma_0}{8\epsilon_0 R^2}; A_4 = 0; A_5 = \frac{-\sigma_0}{16\epsilon_0 R^4}; A_6 = 0 \tag{21}$$

From Eq. (14) we can find the coefficients B_l required for the exterior solution.

Problem 3.23: As shown in Lecture 12, the general solution in cylindrical co-ordinates is given by,

$$V(s, \phi) = C - \frac{\lambda}{2\pi\epsilon_0} ln(s) + \sum_{n=1} (A_n s^n + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi)) \tag{22}$$

Problem 3.25: Since the charge density at the surface is $\sigma = asin(5\phi)$, consisting of only one harmonic, it is easy to write down the solution,

$$V(s < R) = A_5 s^5 sin(5\phi); \quad V(s > R) = \frac{B_5}{s^5} sin(5\phi) \tag{23}$$

Continuity of the potential requires that $A_5 R^5 = B_5/R^5$. The condition on the perpendicular electric field, ie.

$$-\frac{\partial V_{above}}{\partial s}|_R + \frac{\partial V_{below}}{\partial s}|_R = \frac{asin(5\phi)}{\epsilon_0} \tag{24}$$

yields,

$$5A_5 R^4 + 5B_5/R^6 = \frac{a}{\epsilon_0} \quad \text{and using } A_5 R^5 = B_5/R^5; \quad A_5 = a/(10\epsilon_0 R^4); B_5 = -aR^6 \tag{25}$$

so the solution is,

$$V(s < R) = \frac{as^5 sin(5\phi)}{10\epsilon_0 R^4}; \qquad V(s > R) = \frac{aR^6 sin(5\phi)}{10\epsilon_0 s^5} \tag{26}$$

Problem 3.27: The dipole moment is found from,

$$\vec{p} = \int \rho(\vec{r}')\vec{r}' d\vec{r}' = \sum_i \vec{r}_i q_i = (3qa - qa)\hat{z}. \tag{27}$$

The potential due to the diplole is,

$$V = k\frac{\vec{p} \cdot \hat{r}}{r^2} = \frac{2kqacos\theta}{r^2} \tag{28}$$

Problem 3.33: To find the electric field of a dipole, we take the gradient of the potential $V = kpcos\theta/r^2$ in spherical polar co-ordinates (the $\hat{\phi}$ component is clearly zero),

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} = \frac{2kpcos\theta}{r^3}\hat{r} + \frac{kpsin\theta}{r^3}\hat{\theta} \tag{29}$$

Using $\hat{z} = \hat{p} = cos\theta\hat{r} - sin\theta\hat{\theta}$ this may be written as,

$$\vec{E} = k\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} \tag{30}$$

Problem 3.41:
a), b) Average field due to a single charge q_i at position $\vec{r}_i$ inside a sphere is given by,

$$\vec{E}_{ave} = \frac{k}{volume}\int \frac{q_i(\vec{r'} - \vec{r_i})}{|\vec{r'} - \vec{r_i}|^3} d\vec{r'} \tag{31}$$

If we define $\rho_i = -q_i/volume$, this is the same as the superposition formula for a uniform sphere of charge with charge density ρ_i. We know the answer to that problem to be $\vec{E}^1_{ave} = \rho_i\vec{r}_i/3\epsilon_0 = -q_i\vec{r}_i/3\tau\epsilon_0$. Using $\tau = 4\pi R^3/3$, we then find that $\vec{E}^1_{ave} = k\vec{p}_1/R^3$.

c) For an arbitrary distribution of charge, we simply add the dipole moments of each charge so that $\vec{p} = \sum \vec{p}_i$.

d) Consider again Eq. (32). When $\vec{r}_i$ is outside the sphere, this looks like the superposition calculation for the uniform sphere with density ρ_i again, but now we use the expression for the electric field outside a uniform sphere, $\vec{E}^1_{ave} = kq_i\vec{r}_i/r_i^3$. The average electric field inside the sphere due to a charge outside is like that at at the center of the sphere. The generalization to an arbitrary number of charges is by superposition $\vec{E}_{ave} = k\sum q_i\vec{r}_i/r_i^3$.

Problem 3.44: (Check potential calculation in b)) Green's reciprocity theorem states that if we solve two problems, one with charge density ρ_1 and electrostatic potential V_1 and the second with ρ_2 and potential V_2, then these solutions obey,

$$\int \rho_1 V_2 d\vec{r} = \int \rho_2 V_1 d\vec{r} \tag{32}$$

where the integrals are taken over all space. In problem a) we want to find the induced charge when a point charge is placed between two grounded, parallel metal sheets. The metal sheets have their normals in the $\hat{x}$ direction and are separated by distance d. The point charge is placed at distance x from the first sheet. To use the reciprocity theorem, we have to set up two situations. One situation is the one we want to solve and the other is supposed to help us and is a situation that we know the solution to. In this problem situation 1 is the problem we want to solve and situation 2 is simply a parallel plate capacitor with one sheet grounded. First consider that the first sheet is grounded and the second sheet is at potential V_0. In that case $V_2 = V_0x/d$, $Q_2 = \epsilon_0 AV_0/d$, and $\rho_2 = -Q_0\delta(x) + Q_0\delta(x-d)$. In the real problem (situation 1), we can write the charge as $Q_a\delta(x') + q\delta(x-x') + Q_b\delta(x'-d)$ and we want to find Q_a and Q_b. We then have,

$$\int \rho_1 V_2 d\vec{r} = qV_0x/d + Q_bV_0 = \int \rho_2 V_1 dvecr = 0 \quad \text{so} \quad Q_b = -qx/d \tag{33}$$

In a similar way if we now place the charge in the real situation at position $d - x$, and follow the same reasoning we find $Q_a = (x-d)q/d$.

b) Now make a similar construction for two concentric spherical sheels with a charge between them. The first shell has radius a, the second has radius b and the charge is at location r. Both shells are grounded so their potentials are zero. The second situation is just the two shells, with only the first shell (radius a) grounded. The potential of the second shell is V_0. The potentials between the shells is sphericall symmetric and has the form $V(r) = A + B/r$. Using $V(a) = 0$ and $V(b) = V_0$, we find that the potential is given by, $V(r) = [V_0b/(b-a)][1 - a/r]$. Now we use the reciprocity theorem with situation 1 having, $\rho_1 = Q_a\delta(r'-a) + q\delta(r'-r) + Q_b\delta(r'-b)$, and $V_1(a) = V_1(b) = 0$. We do not know the charges in situation 2, but we know the potential, $V_2(r) = [V_0b/(b-a)][1-a/r]$. Using the reciprocity relation we find that,

$$\int \rho_1 V_2 d\vec{r} = qV_0b(1-a/r)/(b-a) + Q_bV_0 = \int \rho_2 V_1 d\vec{r} = 0 \quad \text{so} \quad Q_b = -qb(1-a/r)/(b-a) \tag{34}$$

A similar calculation with the outer shell grounded yields, $Q_a = -qa(b/r-1)/(b-a)$.

Problem 3.49:
We take the angle of the pendulum to the vertical to be $\alpha = \pi - \theta$, where θ is usual angle to the $\hat{z}$ in spherical polar co-ordinates. The force on a pendulum balances the centripital acceleration so that,

$$\vec{F}_{pendulum} = -T\hat{r} - mg\hat{z} = mv^2/r \tag{35}$$

In this problem we are told to use a pendulum released from the horizontal position so that we can relate the kinetic energy to the angle through,

$$mgh = mgrcos\alpha = mv^2/2 \tag{36}$$

Combining these equations we have,

$$T - mgcos\alpha = 2mgcos\alpha, \tag{37}$$

so that $T = 3mgcos\alpha = -3mgcos\theta$. The force on the pendulum, is then,

$$\vec{F}_{pendulum} = -T\hat{r} - mg\hat{z} = mg[3cos\theta\hat{r} - \hat{z}] \tag{38}$$

Now compare this to the force on a charge in the field of a dipole centered at the origin and oriented along the $\hat{z}$ direction. Using $\vec{F} = q\vec{E}$ and the expression for the electric field of a dipole, we have,

$$\vec{F}_q = \frac{kq}{r^3}[3pcos\theta\hat{r} - p\hat{z}] \tag{39}$$

The two forces are the same provided we use the relation $mg = kqp/r^3$.