Outline of solutions to problem set 4

Problem 5.3: a) The electric and magnetic fields are perpendicular and they are constant. Take $\vec{E} = E_0\hat{z}$ and $\vec{B} = B_0\hat{x}$ and the initial velocity to be $v_0\hat{y}$. The initial force is then,

$$\vec{F}_0 = q(E_0\hat{z} + v_0B_0\hat{y}\wedge\hat{x}) = q(E_0 - v_0B_0)\hat{z} \tag{1}$$

Clearly there is no force on the charged particle, and hence the velocity remains the same, provided we choose the velocity selector condition $v_0 = E_0/B_0$.

b) With the electric field switched off, the particle undergoes circular motion, with,

$$\frac{mv_0^2}{R} = qv_0B_0 \quad \text{so that} \quad R = \frac{m}{qB_0}v_0 = \frac{m}{q}\frac{E_0}{B_0^2} \tag{2}$$

The charge to mass ratio is then found by measuring R, E_0 and B_0 to find,

$$\frac{q}{m} = \frac{E_0}{RB_0^2} \tag{3}$$

Problem 5.6: a) The current per unit length $K(s)$ is found from

$$K(s) = \frac{current}{length} = \frac{1}{length}\frac{dQ}{dt} = \frac{1}{ds}\frac{\sigma s ds d\phi}{dt} = \sigma s\frac{d\phi}{dt} = \sigma s\omega \tag{4}$$

b) In this case we want the current per unit area $\vec{J}(r,\theta) = J(r,\theta)\hat{\phi}$. We have $dQ = \rho d\tau = \rho r^2 sin\theta d\phi d\theta dr$ and area $da = rd\theta dr$, so that

$$J(r,\theta) = \frac{1}{area}\frac{dQ}{dt} = \frac{1}{rd\theta dr}\frac{\rho r^2 sin\theta d\phi d\theta dr}{dt} = \rho r\omega sin\theta \tag{5}$$

Problem 5.8: The magnetic field at the center of a symmetric polygonal loop is directed along the loop normal. Take the normal to be the $\hat{z}$ direction and the current to flow counter-clockwise. The magnetic field is then in the positive $\hat{z}$ direction. We use the Biot-Savart law for a wire segment of length $2R$,

$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{l}\wedge\vec{r}}{r^3} \tag{6}$$

We choose the wire to lie on the x-axis, so that the center of the loop is at $R\hat{y}$. In that case $d\vec{l}\wedge\vec{r} = rdx sin\theta\hat{z} = Rdx\hat{z}$, so that,

$$B_z = \frac{\mu_0 i}{4\pi}\int_{-R}^{R}\frac{Rdx}{r^3} = \frac{\mu_0 iR}{4\pi}\int_{-R}^{R}\frac{dx}{(R^2+x^2)^{3/2}} \tag{7}$$

Doing the integral yields,

$$\frac{\mu_0 iR}{4\pi}\frac{x}{R^2(R^2+x^2)^{1/2}}|_{-R}^{R} = \frac{\mu_0 i}{2\sqrt{2}\,\pi R} \tag{8}$$

The field for a square is four times this value as each side makes the same contribution. For a polygon with n sides, with the distance to the center from the center of a segment set to R, the angle subtended by each segment is $2\pi/n$. The length, $2l$ of each segment is found from $tan(\pi/n) = l/R$. The field due to each segment is then the same as that above, but with the limits of integration $-l, l$, so that for one side of the polygon,

$$B_z = \frac{\mu_0 iR}{4\pi}\frac{x}{R^2(R^2+x^2)^{1/2}}|_{-l}^{l} = \frac{\mu_0 i}{2\pi R}\frac{l}{(R^2+l^2)^{1/2}} \tag{9}$$

The total contribution of all sides is n times this value. As $n \to \infty$ we can use the leading order expansion for $tan(2\pi/n) \to 2\pi/n$, so that $2l \approx 2\pi R/n$. Using this expression for l and neglecting l in the denominator, we find that,

$$B_z^n = n\frac{\mu_0 i}{2\pi R}\frac{l}{(R^2+l^2)^{1/2}} \to n\frac{\mu_0 i}{2\pi R}\frac{\pi R}{n}\frac{1}{R} = \frac{\mu_0 i}{2R} \tag{10}$$

which is result for the center of a circular loop.

Problem 5.9: In this problem we use superposition using the magnetic field at the center of a circular loop is $\frac{\mu_0 i}{2R}$ and the field near an infinite wire $\mu_0 i/2\pi r$. In situation a) the straight segments make no contribution as $d\vec{l} \wedge \vec{r}$=0. The contributions of the two circular sections are added to find,

$$B_z = \frac{\mu_0 i}{8}[\frac{1}{a} - \frac{1}{b}] \tag{11}$$

In situation b) the two straight segments add to have the same effect as an infinite wire, so that,

$$B_z = \frac{\mu_0 i}{4R} + \frac{\mu_0 i}{2\pi R} \tag{12}$$

Problem 5.10: a) In Fig. 5.24a) of Griffiths, the magnetic field through the loop is out of the page. The force on the two segments that are normal to the current carrying wire are equal an opposite. The force on the two segments that are parallel to the long wire and opposite but not equal. Using $\vec{F} = i\vec{l} \wedge \vec{B}$ for each of the these wires, with $B = \mu_0 i/(2\pi s)$ for the magnetic field of the long wire, we find the total force on the loop to be,

$$F_{loop} = \frac{\mu_0 I^2 a}{2\pi}[\frac{1}{s} - \frac{1}{s+a}] \quad \text{away from the wire} \tag{13}$$

b) In Fig. 5.24 b) For each segment in the loop, the direction of the forces is perpendicular to the segment in question and toward the center of the triangle. The two sides that are not parallel to the long wire have a parallel component that cancels and a perpendicular component that we need to calculate. The force on the bottom edge of the triangle is the same as that calculated in a), while the vertical component of the force on the slanted edges is found by integrating the vertical component, so that,

$$F_{loop} = \frac{\mu_0 I^2 a}{2\pi}\frac{1}{s} - 2\int_0^a idt cos(\pi/3)\frac{\mu_0 i}{2\pi(s + tcos(\pi/6))} \tag{14}$$

which gives,

$$\vec{F}_{loop} = [\frac{\mu_0 I^2 a}{2\pi}\frac{1}{s} - \frac{\mu_0 I^2}{\sqrt{3}\pi} ln(1 + \frac{\sqrt{3}a}{2s})]\hat{y} \tag{15}$$

Problem 5.13: a) Outside the wire, drawn a circular Amperian loop to enclose the current and it is clear that $\vec{B}(s) = \frac{\mu_0 i\hat{\phi}}{2\pi s}$. Inside the wire a circular Amperian loop has no enclosed current, so the magnetic field is zero inside the cylinder. b) Outside the loop the field is the same as in a). Inside the loop we have $j(s) = As$, with

$$\int jda = \int_0^{2\pi}\int_0^a sd\phi ds As = 2\pi\frac{a^3}{3}A = I \tag{16}$$

Solving we find $A = 3I/(2\pi a^3)$. The current enclosed at location s is then,

$$i(s) = 2\pi\int_0^s s' j(s')ds' = 2\pi A\frac{s^3}{3} = I\frac{s^3}{a^3} \tag{17}$$

The magnetic field is found from Ampere's law,

$$\oint \vec{B}\cdot\vec{l} = 2\pi s B(s) = \mu_0 i(s) = \mu_0 I\frac{s^3}{a^3} \quad \text{so that} \quad B(s) = \mu_0 I\frac{s^2}{2\pi a^3} \tag{18}$$

At $s = a$ the results for inside and outside are the same, as they should be for this case.

Problem 5.15: The magnetic field inside a solenoid is $\mu_0 ni$, while the field outside is zero. We can also use superposition. The inner solenoid has a field to the left on its interior, while the outer solenoid has a field to the right on its interior. (i) In the inner region of the inner solenoid, both solenoids make a contribution, so the field is $B = \mu_0 I(n_1 - n_2)$ to the left. (ii) Between the two solenoids only the outer solenoid contributes, so $B = \mu_0 I n_2$ to the right. (iii) Outside the larger solenoid there is no magnetic field (provided the solenoids are infinitely long!).

Problem 5.24: To show that $\vec{A} = \frac{-1}{2}(\vec{r} \wedge \vec{B})$ corresponds to a constant magnetic field, we have to evaluate $\vec{\nabla} \wedge \vec{A}$. using the vector product rule 8 on the inside cover of Griffiths, we then have,

$$\vec{\nabla} \wedge (\vec{r} \wedge \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{r} - (\vec{r} \cdot \vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{r}). \tag{19}$$

The third term is zero as $\vec{\nabla} \cdot \vec{B} = 0$. Evaluating the first and last terms, we find,

$$\vec{\nabla} \wedge (\vec{r} \wedge \vec{B}) = \vec{B} - (\vec{r} \cdot \vec{\nabla})\vec{B} - 3\vec{B} \tag{20}$$

The second term is zero only when $\vec{B}$ is constant, in which case $\frac{-1}{2}(\vec{r} \wedge \vec{B}) = \vec{B}$ as required.

Problem 5.25: a) We want to find the vector potential for a wire carrying current I. There are many ways to approach this. We can find the magnetic field first and then integrate or we can find the vector potential directly. Lets do it directly using the superposition formula for the vector potential.

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')d\vec{r}'}{|\vec{r} - \vec{r}'|} \tag{21}$$

We choose the wire to lie along the $\hat{z}$ direction so the only component of $\vec{A}$ that is finite is A_z. We then have,

$$A_z = \frac{\mu_0 I}{4\pi} \int_{-l}^{l} \frac{dx}{(s^2 + x^2)^{1/2}} = \frac{\mu_0 I}{4\pi} ln[\frac{l + (l^2 + s^2)^{1/2}}{-l + (l^2 + s^2)^{1/2}}] \tag{22}$$

We want the limit $l \to \infty$, so we can expand this expression to leading order is s^2/l^2 to find,

$$A_z = \frac{\mu_0 I}{4\pi} ln[\frac{2}{(s^2/2l^2)}] \to -\frac{\mu_0 I}{2\pi} ln(s) + constant. \tag{23}$$

Recall that for a line charge the electrostatic potential is $-\lambda ln(s)/(2\pi\epsilon_0) + constant$, ie the same form. The magnetic potential is however a vector $\vec{A} = -\frac{\mu_0 i}{2\pi} ln(s)\hat{z}$. It is easy to see that $\vec{\nabla} \cdot \vec{A} = 0$ as required. Now lets see if we can find this result by integration of the magnetic field, $\vec{B} = \frac{\mu_0 i \hat{\phi}}{2\pi s}$. We know that $\vec{B} = \vec{\nabla} \wedge \vec{A}$ so using the $\hat{\phi}$ component of $\vec{\nabla} \wedge \vec{A}$ in cylindrical co-ordinates, we find,

$$[\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}] = \frac{\mu_0 i}{2\pi s} \tag{24}$$

From the superposition formula, since the current is only in the z-direction, we know that $A_s = 0$, therefore we have,

$$A_z = -\int_a^s ds' \frac{\mu_0 i}{2\pi s'} = -\frac{\mu_0 i}{2\pi} ln(s) + constant \tag{25}$$

b) It is easiest to solve this problem using the integration method. The only difference is that we have to take the expression for the magnetic field inside and outside the wire. Inside the wire the expression found using Ampere's law is,

$$2\pi s B(s) = \mu_0 i(s) = \mu_0 I \frac{s^2}{R^2} \quad \text{so that} \quad B(s) = \frac{\mu_0 i s}{2\pi R^2} \tag{26}$$

Now we integrate the potential in two parts,

$$A_z(s < R) = -\int_a^R ds' \frac{\mu_0 i}{2\pi s'} - \int_R^s ds' \frac{\mu_0 i s'}{2\pi R^2} = -\frac{\mu_0 i s^2}{4\pi R^2} + constant \tag{27}$$

If we want to set the potential at the surface of the wire to be zero, we can use find constants to achieve the forms,

$$\vec{A}(s > R) = -\frac{\mu_0 i}{2\pi} ln(s/R)\hat{z}; \quad \vec{A}(s < R) = -\frac{\mu_0 i}{4\pi}[\frac{s^2}{R^2} - 1]\hat{z} \tag{28}$$

Problem 5.29: We are given the vector potentials for a spherical shell of charge density σ rotating with angular frequency ω,

$$\vec{A}(r < R) = \frac{\mu_0 R\omega\sigma}{3} r sin\theta\hat{\phi}; \quad \vec{A}(r > R) = \frac{\mu_0 R^4 \omega\sigma}{3r^2} sin\theta\hat{\phi} \tag{29}$$

First we need to use this result to find the vector potential for a uniform sphere of charge density $\rho = 3Q/(4\pi R^3)$. We do this by integrating the results above.

$$\vec{A}(r) = \frac{\mu_0 \omega \rho}{3} sin\theta\hat{\phi}[\frac{1}{r^2}\int_0^r r'^4 dr' + r\int_r^R dr'r'] = \frac{\mu_0\omega\rho}{2} r sin\theta[\frac{R^2}{3} - \frac{r^2}{5}]\hat{\phi} \tag{30}$$

To find the magnetic field $\vec{B} = \vec{\nabla} \wedge \vec{A}$. We use the curl in spherical polars and take advantage of the fact that only A_ϕ is finite, so that,

$$\vec{B} = \frac{1}{rsin\theta}\frac{\partial}{\partial\theta}(sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} \tag{31}$$

which gives,

$$\vec{B} = \mu_0\omega\rho[(\frac{R^2}{3} - \frac{r^2}{5})cos\theta\hat{r} - (\frac{R^2}{3} - \frac{2r^2}{5})sin\theta\hat{\theta} \tag{32}$$

Problem 5.33: Here we can use the analogy with an electric dipole, where the magnetic dipole moment replaces the electric dipole moment. The magnetic dipole moment of a current ring is $m = \int ida$, with the direction given by the RHR. However we can also start with the vector potential of a dipole,

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\wedge\hat{r}}{r^2} \rightarrow \frac{\mu_0}{4\pi}\frac{msin\theta}{r^2}\hat{\phi} \tag{33}$$

where the explicit expression is for a dipole moment $\vec{m}$ aligned along the $\hat{z}$ direction. Using the cross product in polar co-ordinates,

$$\vec{B} = \frac{1}{rsin\theta}\frac{\partial}{\partial\theta}(sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta} \tag{34}$$

leads to,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2mcos\theta\hat{r} + msin\theta\hat{\theta}}{r^3} \tag{35}$$

using $\vec{m} = m\hat{z}$ and $\hat{z} = cos\theta\hat{r} - sin\theta\hat{\theta}$ leads to the expression quoted.

Problem 5.37: We use the Biot-Savart rule for each of the four parts of the loop. We did the special case of $z = 0$ in Problem 5.8. Now we need to generalize the expression to any value of z. This is relatively easy as we can use the result in 5.8, we only need to take the right component of it, namely,

$$B_z(z) = \frac{\mu_0 i}{2\pi R}\frac{l}{(R^2 + l^2)^{1/2}} sin\alpha \tag{36}$$

where $l \rightarrow w/2$, $R \rightarrow ((w/2)^2 + z^2)^{1/2}$ and $sin\alpha = w/(2R)$, so that,

$$B_z(z) = \frac{\mu_0 i w^2}{2\pi(z^2 + (w/2)^2)(z^2 + 2(w/2)^2)^{1/2}} \tag{37}$$

Note that the key observation in this problem is that the angle α is the same for all dx when integrating from $-w/2$ to $w/2$. This is because the vectors $d\vec{l}$ and $\hat{r}$ define the same plane for all x on the integration interval, so their cross product is the same for all x.

In the limit of large z, we have,

$$B_z(z) = \frac{\mu_0 i w^2}{2\pi z^3} = \frac{\mu_0 m}{2\pi z^3} \tag{38}$$

where $m = ia$ is the dipole moment. From Problem 5.33 above we have,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{2mcos\theta\hat{r} + msin\theta\hat{\theta}}{r^3} \tag{39}$$

For $r \rightarrow z$ and $\theta = 0$ this reduces to the equation above so it is the correct form for a dipole.