Outline of solutions to Homework 5

Problem 7.1: a) We set the voltage of the inner spherical shell to be V and the potential of the outer one to be 0, so the voltage difference is V as required. By symmetry, the current density is directed radially outward. If I is the total current, then the current density is

$$\vec{j}(r) = \frac{I}{4\pi r^2}\hat{r}. \tag{1}$$

Inside the conductor the voltage obeys Laplace's equation. The system is spherically symmetric so the solution to Laplace's equation in spherical polars reduces to just the $l = 0$ term, ie, $V(r) = A_0 + B_0/r$. We then have (assuming no polarization effects),

$$A_0 + B_0/a = V; \quad A_0 + B_0/b = 0 \quad \text{so that} \quad B_0 = \frac{abV}{b-a}; \quad A_0 = \frac{-aV}{b-a} \quad \text{and} \quad V = \frac{aV}{b-a}[\frac{b}{r} - 1] \tag{2}$$

The electric field is then,

$$\vec{E}(r) = -\frac{\partial V}{\partial r} = \frac{aV}{b-a}[\frac{b}{r^2}]\hat{r} \tag{3}$$

Using $\vec{j} = \sigma\vec{E}$, we find that the magnitude of the $\vec{j}$ is,

$$j(r) = \sigma E(r) = \sigma\frac{aV}{b-a}[\frac{b}{r^2}] = \frac{I}{4\pi r^2} \quad \text{so that} \quad I = 4\pi\sigma\frac{abV}{b-a} \tag{4}$$

b) Using $V = IR$, we then find $R = \frac{b-a}{4\pi\sigma ab}$. This can also be derived by using the series combination of shells,

$$R = \int_a^b \frac{\rho dr}{4\pi r^2} = \frac{b-a}{4\pi\sigma ab} \tag{5}$$

c) In the limit of large b, the resistance is $R = \frac{2}{4\pi\sigma a}$, so the current is $I = V/R = 2\pi\sigma aV$.

Problem 7.2: a) Summing the voltages around the loop, we have,

$$V_C + V_R = 0 = \frac{Q}{C} + iR = \frac{Q}{C} + R\frac{dQ}{dt} \quad \text{so that} \quad Q(t) = Q_0 e^{-t/RC} \tag{6}$$

where $Q_0 = CV_0$. The current is

$$i = -\frac{dQ}{dt} = (Q_0/RC)e^{-t/RC} = (V_0/R)e^{-t/RC} \tag{7}$$

b) The energy stored in the capacitor is $CV_0^2/2$, while the energy found from the dissipated power is,

$$U = \int_0^\infty i^2 R dt = \int_0^\infty (\frac{V_0}{R})^2 e^{-2t/RC} dt = (\frac{-RC}{2})(\frac{V_0}{R})^2|_0^\infty = \frac{1}{2}CV_0^2 \tag{8}$$

c) The circuit equation is now,

$$V_0 + V_C + V_R = 0 = V_0 - \frac{Q}{C} - iR = V_0 - \frac{Q}{C} - R\frac{dQ}{dt} \quad \text{so that} \quad Q(t) = Q_0(1 - e^{-t/RC}) \tag{9}$$

The current is $dQ/dt = (V_0/R)e^{-t/RC}$. The total energy supplied by the battery is,

$$\int_0^\infty V_0 I dt = (\frac{V_0^2}{R}) \int_0^\infty e^{-t/RC} dt = CV_0^2 \tag{10}$$

The energy dissipated in the resistor is the same as that calculated for the discharge case. Therefore one half of the power is dissipated in the resistor, no matter how small or large the resistance is.

Problem 7.7: We have a metal bar on conducting rails moving through a region of constant magnetic field into the page. The loop has resistance R. a) The motional emf is Blv, so the current is Blv/R upward in the bar in Fig.

7.16. b) The magnetic drag force is ilB, opposite to the direction of motion. $ilB = B^2l^2v/R$ c) Newton's second law gives,

$$F_{drag} = -\frac{B^2l^2}{R}v = m\frac{dv}{dt} \quad \text{so that} \quad v(t) = v_0 e^{-\alpha t} \tag{11}$$

where $\alpha = \frac{B^2l^2}{mR}$. c) The energy delivered to the resistor is,

$$U = \int i^2(t)Rdt = \frac{(Bl)^2}{R}\int_0^\infty v^2 dt = \frac{(Blv_0)^2}{R}\int_0^\infty e^{-2\alpha t}dt = \frac{1}{2}mv_0^2 \tag{12}$$

Problem 7.8: a) The magnetic field due to the wire is $B(r) = \mu_0 i/(2\pi s)$ out of the page. The flux through the square loop is,

$$\phi_B = \int Bdsdz = a\int_s^{s+a} \frac{\mu_0 i}{2\pi s} = \frac{\mu_0 ia}{2\pi} ln(1 + a/s) \tag{13}$$

b) When the loop is pulled away from wire, the flux (out of page) decreases. Current is induced to increase the flux (out of page) in the loop. The induced current in the loop is then counterclockwise. The emf generated is

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}[\frac{\mu_0 ia}{2\pi} ln(1 + a/s)] = \frac{\mu_0 ia}{2\pi}\frac{s}{a+s}\frac{a}{s^2}\frac{ds}{dt} \tag{14}$$

Using $v = ds/dt$, this reduces to

$$\mathcal{E} = \frac{\mu_0 i}{2\pi}\frac{a^2 v}{s(a+s)} \tag{15}$$

c) If the loop is pulled to the right, the flux through the loop remains constant so there is no induced emf.

Problem 7.11: The motional emf is Blv and the magnetic drag force is $(Bl)^2v/R$ upward. Newton's second law is then

$$mg - F_{drag} = m\frac{dv}{dt} \quad \text{or} \quad \frac{dv}{dt} + \frac{(Bl)^2}{mR}v - g = 0 \tag{16}$$

The resistance of the loop is $R = 4l\rho_r/a$, where ρ_r is the resistivity of aluminum, l is the total perimeter of the loop and a is the crossectional area of the wires making up the loop. The mass of the loop is $4al\rho_m$, where ρ_m is the density of aluminum ($2700kg/m^3$). Therefore,

$$\frac{(Bl)^2}{mR} = \frac{(Bl)^2 a}{a16l^2\rho_r\rho_m} = \frac{B^2}{16\rho_m\rho_r} = b \tag{17}$$

Solving the equation we have

$$v(t) = v_\infty(1 - e^{-bt}) \quad \text{where} \quad v_\infty = g/b = \frac{16g\rho_m\rho_r}{B^2} \tag{18}$$

b is the inverse of the relaxation time. For $B = 1T$, $g = 9.8m/s^2$, $\rho_m = 2700kg/m^3$ and conductivity $\sigma = 1/\rho_r = 1/(2.65 * 10^{-8})(\Omega m)^{-1}$. Plugging the numbers gives $v_\infty = 1.12cm/s$. The relaxation time is $1/b \approx 1.15ms$. This is the time require to reach $1 - 1/e$ of the terminal velocity (about 63%). If the loops were cut, there could be no current and hence no magnetic drag force. The loop would fall under gravity in the usual way.

Problem 7.13: Square loop of side a in the first quadrant of xy plane, with field $\vec{B} = ky^3t^2\hat{z}$. The magnetic flux through the loop is,

$$\phi_B = akt^2\int_0^a y^3 dy = \frac{1}{4}ka^5t^2 \tag{19}$$

The magnitude of the induced emf is $d\phi_B/dt = ka^5t/2$. For positive k, The flux is increasing in the positive $\hat{z}$ direction, so the induced flux opposes this increase. Therefore for positive k the induced emf in the loop is clockwise in the x-y plane.

Problem 7.17: a) The magnetic field in the solenoid is $B = nI\mu_0$, so the flux in the solenoid is $\phi_B = ni\mu_0\pi a^2$. The magnitude of the induced emf in the loop is then $nk\mu_0\pi a^2$. The flux through the solenoid is to the right and increasing (for k positive), so the induced current in the loop is opposite that of the solenoid. The induced current then flows to the right through the resistor and has magnitude $nk\mu_0\pi a^2/R$. b) The total charge that flows is

$$Q = \int_{-\infty}^{\infty} idt = \int_{-\infty}^{\infty} \frac{1}{R}\frac{d\phi_B}{dt}dt = \frac{2\phi_B}{R} = \frac{2nI\mu_0\pi a^2}{R} \tag{20}$$

Problem 7.20: Definition of mutual inductance: $\phi_2 = M_{21}i_1$ (when $i_2 = 0$), $\phi_1 = M_{12}i_2$ (when $i_1 = 0$), Self-inductance is defined through $\phi_B = Li$. a) Assume that the small loop is a long way from the large loop so the magnetic field due to the large loop is essentially constant and is that on the $z - axis$ of a circular ring (see Lecture 18, Eq. 9), then,

$$\vec{B}_{big} = \frac{\mu_0 i_{big}}{2}\frac{s^2}{(s^2+z^2)^{3/2}} \quad \text{so that} \quad \phi_{small} = M_{21}i_{big}, \quad \text{with} \quad M_{21} = \frac{\mu_0 Area_{small}}{2}\frac{b^2}{(b^2+z^2)^{3/2}} \tag{21}$$

where $Area_{small} = \pi a^2$. Now lets find the flux coupling using B_{small} as the source. In that case we can approximate the field as that of a dipole, so that,

$$\vec{B}_{small} = \frac{\mu_0 m}{4\pi r^3}[2cos\theta\hat{r} + sin\theta\hat{\theta}] \tag{22}$$

We can choose a convenient surface to calculate the flux through the big loop - the spherical cap bounded by the big loop, so that,

$$\phi_{big} = \int \vec{B}_{small}\cdot da\hat{r} = \int_0^{2\pi}\int_0^{\theta_{cap}} \frac{\mu_0 im}{4\pi R^3}(2cos\theta)R^2 sin(\theta)d\phi d\theta = \frac{\mu_0 im}{4R}sin^2(\theta_{cap}) \tag{23}$$

where $R^2 = z^2 + b^2$ and $sin(\theta_{cap}) = b/R$. Therefore, This reduces to,

$$\phi_{big} = \frac{\mu_0 i_{small} Area_{small} b^2}{2(z^2+b^2)^{3/2}} = M_{12}i_{small} \tag{24}$$

Comparing Eq. (21) and Eq. (24) we see that $M_{21} = M_{12}$.

Problem 7.23: The self-inductance is found from $\phi_B = Li$. The magnitude of the magnetic field within the loop in Fig. 7.37 is $B(s) = \frac{\mu_0 i}{2\pi}[\frac{1}{s} + \frac{1}{d-s}]$, so the magnetic flux within the loop is,

$$\phi_B = l\int_{\epsilon}^{d-\epsilon} 2\frac{\mu_0 i}{2\pi}\frac{1}{s}ds = \frac{\mu_0 il}{\pi}ln[(d-\epsilon)/\epsilon] \tag{25}$$

This diverges as $\epsilon \to 0$ due to the divergence of the magnetic field. In reality the field does not diverge in a real wire and this divergence is due to the mathematical simplification of a very narrow wire.

Problem 7.27: The field of a toroid with center radius R, and circular cross-section of radius b is found by assuming the the field only depends on r, so that $B_{outside} = 0$. Then the field inside is found from,

$$2\pi rB(|r-R| < b) = \mu_0 Ni \quad \text{so that} \quad B_{inside} = \frac{\mu_0 Ni}{2\pi r} \tag{26}$$

The energy stored in the toroid is,

$$W = \frac{1}{2\mu_0}h\int_a^b 2\pi r(\frac{\mu_0 Ni}{2\pi r})^2 dr = \frac{\mu_0 N^2 i^2}{4\pi}hln(b/a) \tag{27}$$

This work should be equal to the energy found using the formula $Li^2/2$ and with the formula Eq. (7.27), we have,

$$U = \frac{\mu_0 N^2 h}{4\pi}ln(b/a)i^2 \tag{28}$$

demonstrating the equivalence of the two methods.

Problem 7.54 a) The definitions of L_1, L_2, M are,

$$N_1\phi_1 = L_1 i_1 + M i_2; \quad N_2\phi_2 = L_2 i_2 + M i_1; \tag{29}$$

When $i_2 = 0$ we have,

$$N_1\phi_1 = L_1 i_1; \quad N_2\phi_2 = M i_1; \tag{30}$$

while when $i_1 = 0$,

$$N_2\phi_2 = L_2 i_2; \quad N_1\phi_1 = M i_2; \tag{31}$$

Since L_1, L_2, M are constants, we can find them from any of these equations. Eliminating the flux from the latter two sets yields $M i_2 = L_1 i_1; \quad M i_1 = L_2 i_2$ and hence $M^2 = L_1 L_2$.

b) Taking a time derivative of Eqs. (29) above and using both Faraday's law and Kirchhoff's voltage law, we have,

$$V_1 cos(\omega t) = L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt}; \quad -I_2 R = L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} \tag{32}$$

c) Using the second of these equations to eliminate I_1 from the first equation also leads to cancellation of the time derivative term in I_2 in the first equation. We can then solve for

$$I_2(t) = -\frac{V_1 L_2}{MR} cos(\omega t) \tag{33}$$

Substituting this back into the second of equations (32) and solving yields,

$$I_1(t) = \frac{V_1}{L_1}[\frac{1}{\omega} sin(\omega t) + \frac{L_2}{R} cos(\omega t)] \tag{34}$$

d) Taking the ratio $I_2 R/V_{in}$ gives,

$$\frac{-\frac{V_1 L_2}{M} cos(\omega t)}{V_1 cos(\omega t)} = -\frac{L_2}{M} \tag{35}$$

Noting that for an ideal transformer $\phi_1 = \phi_2$ Eq. (31) gives $L_2/M = N_2/N_1$ and hence the result quoted. The minus sign indicates that the input and out are π out of phase.
e) Show that the time average of $V_{in} I_1 = V_{out} I_2 = I_2^2 R$.

$$V_{in} I_1 = \frac{V_1}{L_1}[\frac{1}{\omega} sin(\omega t) + \frac{L_2}{R} cos(\omega t)] V_1 cos(\omega t) \tag{36}$$

The time average corresponds to an average of a period of this function. The average of $sin(x)cos(x)$ over a period is zero, while the time average of $cos^2(x)$ over a period is $T/2$, where T is the period. In our case $\omega = 2\pi/T$, so the average over a period gives, $\frac{\pi}{\omega}$ and hence the time average of,

$$< V_{in} I_1 >_t = \frac{\pi V_1^2 L_2}{L_1 R} \tag{37}$$

We also have,

$$< V_{out} I_2 >_t = < I_2^2 R >_t = < (\frac{V_1 L_2}{MR} cos(\omega t))^2 R >_t = \frac{\pi V_1^2 L_2^2}{M^2 R} = \frac{\pi V_1^2 L_2}{L_1 R} \tag{38}$$

Problem 9.9 a) A wave polarized in the $z - direction$ and travelling in the negative $x - direction$ has electric field, $\vec{E} = E_0 \hat{z} cos(kx + \omega t)$ and its propagation vector is $\vec{k} = -k\hat{x}$. It's magnetic field is then

$$\vec{B} = \frac{1}{c}\hat{k} \wedge \vec{E} = \frac{E_0}{c}\hat{y} cos(kx + \omega t) \tag{39}$$

b) In this case $\vec{k} = \frac{k}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$, with $\omega = c|\vec{k}|$.. The polarization is in the $x - z$ plane, so we have $\hat{n} = a\hat{x} + b\hat{z}$. However we know that $\hat{n} \cdot \hat{k} = 0$, so that $a = -b$. Normalization of $\hat{n}$ then yields, $\hat{n} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{z})$. Note that both $\hat{n}$ and $-\hat{n}$ are correct. The electric field is given by,

$$\vec{E} = E_0 \hat{n} cos(\vec{k} \cdot \vec{r} - \omega t) = E_0 \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}) cos(\frac{k}{\sqrt{3}}(x + y + z) - \omega t) \tag{40}$$

The magnetic field is,

$$\vec{B} = \frac{1}{c}\hat{k} \wedge \vec{E} = \frac{E_0}{c}\frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z}) \wedge \frac{1}{\sqrt{2}}(\hat{x} - \hat{z}) cos(\frac{k}{\sqrt{3}}(x + y + z) - \omega t) \tag{41}$$

which reduces to

$$\vec{B} = \frac{E_0}{\sqrt{6}c}(-\hat{x} + 2\hat{y} - \hat{z}) cos(\frac{k}{\sqrt{3}}(x + y + z) - \omega t) \tag{42}$$