Outline of solutions to Homework 6

Problem 4.22: Since the cylinder is a uniform dielectric, the electrostatic potential obeys Laplace's equation inside and outside. From experience, we know the solution inside corresponds to a uniform electric field, so $V_{in} = -a_1 x = -a_1 s cos\phi$. The solution outside is the sum of a term corresponding to the applied field plus a term of the dipole form in cylindrical systems (i.e. the $n = 1$ term in cylindrical co-ordinate solution of Laplace's equation), so that $V_{out} = -E_0 x + b_1 cos\phi/s = -E_0 s cos\phi + b_1 cos\phi/s$. To find the constants a_1, b_1, we use two boundary conditions at the surface of the cylinder, $V_{in}(a,\phi) = V_{out}(a,\phi)$ and $\epsilon_0 E^{\perp}_{out}(a,\phi) - \epsilon E^{\perp}_{in}(a,\phi) = \sigma_f$. In this problem $\sigma_f = 0$, so we find,

$$-a_1 a = -E_0 a + \frac{b_1}{a}; \quad \epsilon a_1 = \epsilon_0 E_0 + \epsilon_0 \frac{b_1}{a^2} = 0 \tag{1}$$

where $\epsilon = \epsilon_0(1 + \chi_e)$. Solving gives $a_1 = 2E_0/(\chi_e + 2)$; $b_1 = E_0 a^2 \chi_e/(\chi_e + 2)$. The electric field on the interior is $\vec{E}_{in} = -\vec{\nabla} V$. Carrying out this gradient in Cartesian co-ordinates yields $\vec{E}_{in} = a_1 \hat{x} = 2E_0 \hat{x}/(\chi_e + 2)$

Problem 4.26: The energy density is $\epsilon E^2/2$, so we can find the total energy by integration. The electric field in the metal is zero so it does not contribute. The electric field inside the dielectric region is $Q/(4\pi\epsilon r^2)$, while the field outside the dielectric is $Q/(4\pi\epsilon_0 r^2)$. We then have,

$$U = \frac{\epsilon}{2}(\frac{Q}{4\pi\epsilon})^2 \int_a^b 4\pi \frac{r^2 dr}{r^4} + \frac{\epsilon_0}{2}(\frac{Q}{4\pi\epsilon_0})^2 \int_b^\infty 4\pi \frac{r^2 dr}{r^4} = \frac{Q^2}{8\pi\epsilon_0}[\frac{1}{\epsilon_r a} - \frac{1}{\epsilon_r b} + \frac{1}{b}] \tag{2}$$

Problem 4.28: The energy gain when the dielectric oil enters the cylindrical cavity is $[C_{after} - C_{before}]V^2/2$. Since V is fixed, we need the expression for the capacitance of a cylindrical capacitor when part of the cylindrical cavity has dielectric susceptibility χ_e. The capacitance of a cylindrical capacitor uniformly filled with material of dielectric constant ϵ is $C = 2\pi L\epsilon/ln(b/a)$, where L is the length of the cylinder. When the cavity is composed of two parts the two parts act as capacitors in parallel, so the capacitance of the system when it is filled to height h is,

$$C(h) = \frac{2\pi\epsilon_0}{ln(b/a)}[h\epsilon_r + L - h] \tag{3}$$

The energy gain when the oil fills the cavity is then $(C(h) - C(0))V^2/2$ which reduces to,

$$\delta U_{cap} = \frac{\pi h \epsilon_0 \chi_e}{ln(b/a)} V^2 \tag{4}$$

where we used $\chi_e = \epsilon_r - 1$. The energy cost of raising the oil to height h is,

$$\delta U_{grav} = \pi(b^2 - a^2) g\rho \int_0^h h' dh' = \frac{1}{2}\pi\rho g h^2 (b^2 - a^2) \tag{5}$$

The total energy change when the oil enters the cylindrical cavity to height h is then $U = \delta U_{grav} - \delta U_{cap}$. The equilibrium condition is $-\vec{\nabla} U = 0$ which sets the force to zero. In this problem the gradient reduces to a derivative with respect to h. We then find the equilibrium condition to be,

$$h = \frac{\epsilon_0 \chi_e V^2}{\rho g (b^2 - a^2) ln(b/a)} \tag{6}$$

Problem 4.32 Using Gauss's law for the displacement field $\vec{D}$ and the relations $\vec{D} = \epsilon \vec{E}$ and $\vec{P} = \epsilon_0 \chi_e \vec{E}$, we find,

$$\vec{D} = \frac{q\hat{r}}{4\pi r^2}; \quad \vec{E} = \frac{q\hat{r}}{4\pi\epsilon r^2} \quad \vec{P} = \frac{q\chi_e \hat{r}}{4\pi(1 + \chi_e) r^2} \tag{7}$$

The bound charge densities are

$$\sigma_b = \hat{n} \cdot \vec{P} = \hat{r} \cdot \vec{P} = \frac{q\chi_e}{4\pi(1 + \chi_e) r^2} \tag{8}$$

and

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = 0 \quad r > 0 \tag{9}$$

The total bound charge on the surface is

$$q_b(R) = \int_0^{\pi}\int_0^{2\pi} R^2 d\theta d\phi \frac{q\chi_e}{4\pi(1+\chi_e)R^2} = \frac{q\chi_e}{(1+\chi_e)} \tag{10}$$

The compensating bound charge is next to the point charge and may be written $-\frac{q\chi_e}{(1+\chi_e)}\delta^3(\vec{r})$.

Problem 4.34 The potential has a dipole form outside the sphere and has the form of a dipole plus a constant electric field (along the dipole axis) inside the sphere. We then take,

$$V_{in} = (a_1 r + \frac{p}{4\pi\epsilon r^2})cos\theta; \quad V_{out} = \frac{b_1}{r^2}cos\theta \tag{11}$$

The potential has to be continuous across the sphere surface and there is no free charge at the surface so we must also have $\epsilon_0 E^{\perp}_{out} - \epsilon E^{\perp}_{in} = 0$. These equations yield,

$$a_1 R + \frac{p}{4\pi\epsilon R^2} = \frac{b_1}{R^2}; \quad \epsilon_0\frac{-2b_1}{R^3} = \epsilon(a_1 - \frac{2p}{4\pi\epsilon R^3}) \tag{12}$$

Solving yields $b_1 = \frac{3p}{4\pi(\epsilon+2\epsilon_0)}$ and $a_1 = \frac{p}{4\pi R^2\epsilon_0}[\frac{3}{\epsilon_r+2} - \frac{1}{\epsilon_r}]$. which yield the solution given.

Problem 4.40 When an electric field is applied to a polar molecule, two things can happen, there is an induced polarization as occurs for non-polar molecules and there is tendency toward alignment of the permanent dipoles in the molecules with the applied field due to the energy gain $U = -\vec{p}\cdot\vec{E}$. In small polar molecules, the permanent dipole term dominates and the polarization effect of an applied electric field is due to the alignment of the permanent dipoles with the electric field rather than the induced polarization. In this case we need to use statistical physics to calculate the polarization effect. This calculation proceeds by assuming that initially we have a set of dipoles of magnitude p_0, but that are randomly oriented due to thermal disorder, and there is no macroscopic polarization. When an electric field is applied, the dipoles can reduce their energy by aligning with the electric field. We calculate the average polarization by using statistical physics to average over all orientations of the molecules, so that for an electric field in the z direction,

$$< p_z > = \frac{\int_0^{2\pi}\int_0^{\pi} e^{-U/k_BT} p_0 cos\theta d\phi p_0^2 sin\theta d\theta}{\int_0^{2\pi}\int_0^{\pi} e^{-U/k_BT} d\phi p_0^2 sin\theta d\theta} \tag{13}$$

or

$$< p_z > = \frac{\int_0^{2\pi}\int_0^{\pi} e^{p_0 E cos\theta/k_BT} p_0 cos\theta d\phi sin\theta d\theta}{\int_0^{2\pi}\int_0^{\pi} e^{p_0 E cos\theta/k_BT} d\phi sin\theta d\theta} \tag{14}$$

Doing the ϕ integral and using $u = cos\theta$, this reduces to,

$$< p_z > = p_0\frac{\int_{-1}^{1} du e^{p_0 E u/k_BT} u du}{\int_{-1}^{1} du e^{p_0 E u/k_BT} du} = \frac{p_0}{a}\frac{e^{au}(-1+au)|_{-1}^{1}}{e^{au}|_{-1}^{1}} = p_0[-\frac{1}{a} + coth(a)] \tag{15}$$

where we used $\int e^{au} u du = e^{-au}(-1+au)/a^2$, and $a = p_0 E/k_B T$. This is called the Langevin formula and is a non-linear relation between the applied electric field and the polarization.

b) The Langevin formula is non-linear, but when a is small, we can carry out a systematic expansion using $Coth(a) = \frac{1}{a} + a/3 + ...$ so that,

$$< p_z > = p_0[-\frac{1}{a} + coth(a)] \approx \frac{p_0 a}{3} = \frac{p_0^2 E}{3k_B T} \tag{16}$$

This is linear in a so the response is linear at high temperatures or low electric fields.

Problem 6.5: The current density is $\vec{j} = j_0\hat{z}$ for all y, z and $-a < x < a$. By symmetry, the magnetic field is in the $\hat{y}$ direction. Using an amperian contour, we find that $2Hw = i_f = 2j_0 xw$, so that $H(|x| < a) = j_0 x\hat{y}$ and hence $\vec{B} = \mu_0\vec{H}$. The force on the dipole is found from $\vec{F} = -\vec{\nabla}U$, where the energy of a dipole in a field is $U = -\vec{m}\cdot\vec{B}$.

a) In this case

$$\vec{F} = \vec{\nabla}(m_0\hat{x}\cdot\mu_0 j_0 x\hat{y}) = 0 \tag{17}$$

b) In this case

$$\vec{F} = \vec{\nabla}(m_0\hat{x}\cdot\mu_0 j_0 x\hat{x}) = \hat{x}\frac{\partial}{\partial x}m_0\mu_0 j_0 x = \mu_0 m_0 j_0\hat{x} \tag{18}$$

c) Use product rule number 4 from Griffiths, so that

$$\vec{\nabla}(\vec{p}\cdot\vec{E}) = \vec{p}\wedge(\vec{\nabla}\wedge\vec{E}) + \vec{E}\wedge(\vec{\nabla}\wedge\vec{p}) + (\vec{p}\cdot\vec{\nabla})\vec{E} + (\vec{E}\cdot\vec{\nabla})\vec{p} \tag{19}$$

The second and fourth terms area zero as $\vec{p}$ does not have any spatial dependence. The first term is zero as in electrostatics curl of $\vec{E}$ is zero. Carrying out the analogous expansion in the magnetic case leads to the same conclusion for the second and fourth terms, but the first term is no longer zero.

Problem 6.6 Al, Cu, $CuCl_2\, Na$ have an odd number of electrons and are expected to be paramagnetic. The remainder have an even number of electrons and are expected to the diamagnetic. This simple rule works for all but Cu which is slightly diamagnetic. It does not work in all cases however as it predicts that $Fe(Z=26)$ should be diamagnetic however it is a ferromagnet - this is due to a quantum mechanical effect call Hund's rule where the electrons in atomic orbitals no longer form up and down spin pairs, but instead from higher spin states.

Problem 6.8 We are given a magnetization $\vec{M} = ks^2\hat{\phi}$ in a long cylinder of radius R with its axis on the $\hat{z}$ axis. The bound surface currents are then $\vec{K}_b = \vec{M}\wedge\hat{n} = -kR^2\hat{z}$, and the bulk bound current density is

$$\vec{j}_b = \vec{\nabla}\wedge\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(s(ks^2))\hat{z} = 3ks\hat{z} \tag{20}$$

Using Ampere's law for a circle around the cylinder axis, we have $2\pi sB(s) = \mu_0 i_b$. We find i_b as follows,

$$i_b(s<R) = \int_0^s (3ks')2\pi s' ds' = 2\pi ks^3; \quad i_b(s>R) = \int_0^R (3ks')2\pi s' ds' + (-kR^2)(2\pi R) = 0 \tag{21}$$

We then find that $B(s>R) = 0$, while $B(s<R) = \mu_0 ks^2\hat{\phi} = \mu_0\vec{M}$.

Problem 6.13: In each of these cases, we are given a magnetization in the material and the shape of a cavity inside the material. The magnetic field, magnetic intensity and magnetization in the material are $\vec{B}_0, \vec{H}_0, \vec{M}$. In the cavity there is no magnetization, the magnetic field is $\vec{B}$ and the magnetic intensity is $\vec{H}$. In the material we have $\vec{B}_0 = \mu_0(\vec{M}+\vec{H}_0)$ while in the cavity we have $\vec{B} = \mu_0\vec{H}$.

We can treat each small cavity using superposition, so the magnetic field inside the cavity is $\vec{B}_0 - \vec{B}_{cavity\ magnetization}$. The magnetic field due to the cavity magnetization can be treated separately by using the results for the field inside a uniformly magnetized region of various shapes. The results are: for a uniformly magnetized sphere $\vec{B}_{in} = \frac{2}{3}\mu_0\vec{M}$; for a uniformly magnetized needly $\vec{B}_{in} = \mu_0\vec{M}$; for a uniformly magnetized disc $\vec{B}_{in} = 0$. We then find that the field and magnetic intensity for the center of the three cavity types is as follows.
a) Sphere:

$$\vec{B} = \vec{B}_0 - \frac{2}{3}\mu_0\vec{M} \quad \vec{H} = \frac{\vec{B}}{\mu_0} = \frac{\vec{B}_0}{\mu_0} - \frac{2}{3}\vec{M} = \vec{H}_0 + \frac{1}{3}\vec{M} \tag{22}$$

b) Needle

$$\vec{B} = \vec{B}_0 - \mu_0\vec{M} \quad \vec{H} = \vec{H}_0 \tag{23}$$

c) Disc

$$\vec{B} = \vec{B}_0 \quad \vec{H} = \vec{H}_0 + \vec{M} \tag{24}$$

Problem 6.18

Take the external field to be $\vec{B}_0 = \mu_0\vec{H}_0 = \mu_0 H_0\hat{z}$. Since there are no free currents we have $\vec{\nabla}\wedge\vec{H} = 0$, so that $\vec{H} = -\vec{\nabla}\phi_m$, where ϕ_m is the magnetic potential. Using $\vec{\nabla}\cdot\vec{H} = 0$ (which applies for a uniform magnetic sphere), we find that $\nabla^2\phi_m = 0$. Since the external field is uniform, we can write a solution that is of the same form as for the analogous electrical case,

$$\phi_m^{in} = -C_0 r cos\theta; \quad \text{and} \quad \phi_m^{out} = -H_0 r cos\theta + C_1\frac{R^3 cos\theta}{r^2} \tag{25}$$

To proceed further we need to know the boundary conditions on $\vec{H}$, $\vec{B}$ and on ϕ_m. We need ϕ_m to be continuous at $r = R$, so that, $-C_0 = C_1 - H_0$. From Ampere's law for $\vec{H}$ we find that at the sphere surface, $H^{\|}_{out} - H^{\|}_{in} = K_f$, where K_f is the free current density at the surface which in this case is zero. Gauss's law for magnetism indicates that $B^{\perp}_{out} = B^{\perp}_{in}$ at the sphere surface. Using this equation we have,

$$-\mu_0 \frac{\partial \phi_m^{out}}{\partial r}|_R = -\mu \frac{\partial \phi_m^{in}}{\partial r}|_R \quad \text{so that} \tag{26}$$

which gives $\mu C_0 = \mu_0(H_0 + 2C_1)$. We then have,

$$C_0 = H_0 - C_1, \quad \text{and} \quad \mu C_0 = \mu_0(H_0 + 2C_1) \quad \text{so that} \quad C_1 = \frac{(1-\mu_r)H_0}{2\mu_r + 1}; C_0 = \frac{3H_0}{\mu_r + 2} \tag{27}$$

The magnetic intensity inside the sphere is $C_0\hat{z}$, so the magnetic field inside is $\mu C_0 \hat{z}$ which gives,

$$\vec{B}_i n = \mu \frac{3H_0}{\mu_r + 2}\hat{z} = \frac{3\mu_r}{\mu_r + 2}\vec{B}_0 = \frac{1+\chi_m}{1+\chi_m/3}\vec{B}_0 \tag{28}$$

Problem 6.25: Assuming that the two donuts can be treated as dipoles, we can write the energy of interaction between the them as,

$$|U| = |\vec{m}_1 \cdot \vec{B}_2| = \frac{\mu_0}{4\pi}\vec{m}_1 \cdot \frac{(3\vec{m}_2 \cdot \hat{r})\hat{r} - \vec{m}_2}{r^3} \rightarrow \frac{\mu_0}{4\pi}\frac{2m^2}{z^3} \tag{29}$$

where we used the fact that both moments and the radial direction are along $\hat{z}$ in this problem. The magnitude of the magnetic force between the donuts is the derivative with respect to "z" of this energy yielding,

$$F_{magnetic} = \frac{3\mu_0}{2\pi}\frac{m^2}{z^4} \tag{30}$$

a) At equilibrium this force is balanced by the gravitational force $F_{grav} = m_d g$. Equating the two forces and solving for z yields,

$$z_{eq} = [\frac{3\mu_0}{2\pi}\frac{m^2}{m_d g}]^{1/4} \tag{31}$$

b) When the third donut is added so that it is attracted to the first one, there are two force equations: for the second donut we have,

$$\frac{3\mu_0}{2\pi}[\frac{m^2}{z_1^4} - \frac{m^2}{z_2^4}] - mg = 0 \tag{32}$$

while for the third donut

$$\frac{3\mu_0}{2\pi}[\frac{-m^2}{(z_1+z_2)^4} + \frac{m^2}{z_2^4}] - mg = 0 \tag{33}$$

These equations need to be solved for z_1 and z_2. To find the ratio of the two heights subtract the two equations and multiply by z_1^4 yielding,

$$1 - 2R^4 + \frac{R^4}{(1+R)^4} = 0 \tag{34}$$

where $R = z_1/z_2$. Solving this numerically yields $R = 0.850115...$