PHY481 - Lecture 10: The image charge method

Griffiths: Chapter 3

A key property of electric fields and potentials in electrostatics is that of uniqueness, which means that if we find a solution that satisifies the boundary conditions, then the solution is unique. The image charge method is a method of "guessing" the solution and then checking to see if the boundary conditions are satisfied. It is a method for solving some problems where a charge or many charges are near a conductor or dielectric material, by finding analogous problems involving only point charges with the conductor or dielectric removed from the problem. Here we study only cases of one point charge, but superposition would enables us to generalize to the case of many charges. A surprising range of problems can be solved in this way, and it is a favorite method for more advanced EM tests as many of these solutions are quite non-trivial and require good physical insight to understand why they work. We shall look at two basic problems: (i) A point charge near a conducting surface and (ii) a point charge near a conducting sphere. These calculations can be generalized to a wide variety of cases, including dielectric materials.

A point charge above a conducting surface.

Consider a system where the region $z \le 0$ is occupied by a grounded conductor, while for $z > 0$ it is vaccuum, with a point charge (q) at position d. We want to find the voltage and electric field in the region $z > 0$ above the conductor, because for $z < 0$ the solution is trivial $\vec{E} = 0$, $V = 0$ (because the conductor is grounded). We also want to find the induced surface charge on the surface of the conductor $\sigma(x, y, 0)$, that screens the electric field from the interior of the conductor, the total screening charge, the force on the charge, and the work required to bring the charge from infinity to the position d.

The boundary conditions are (i) The voltage for $z = 0$ is zero as the conductor is grounded. (ii) the electric field at $z = 0$ is normal to the surface of (in the $\hat{z}$ direction), and $E_z = \sigma/\epsilon_0$. Of course our solution must satisfy the various ways of finding the potential i.e.

$$\nabla^2 V = -\rho/\epsilon_0; \qquad V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r} - \vec{r}_i|}; \qquad V(\vec{r}) = \int \frac{k\rho(\vec{r}')d\vec{r}'}{|\vec{r} - \vec{r}'|} \tag{1}$$

The solution is constructed by noting that we can satisfy the boundary condition $V(x, y, 0) = 0$ with the function,

$$V(\vec{r}) = k[\frac{q}{(x^2 + y^2 + (z - d)^2)^{1/2}} - \frac{q}{(x^2 + y^2 + (z + d)^2)^{1/2}}]. \tag{2}$$

Now lets check that this solution satisfies the other boundary condition $E_x(z = 0) = E_y(z = 0) = 0$. Using $\vec{E} = -\vec{\nabla} V$, we find,

$$E_x = -\frac{\partial V}{\partial x} = kq[\frac{x}{(x^2 + y^2 + (z - d)^2)^{3/2}} - \frac{x}{(x^2 + y^2 + (z + d)^2)^{3/2}} \tag{3}$$

This reduces to zero at $z = 0$ as required. A similar expression is found for E_y, while the electric field in the z-direction is,

$$E_z = -\frac{\partial V}{\partial z} = kq[\frac{z - d}{(x^2 + y^2 + (z - d)^2)^{3/2}} - \frac{z + d}{(x^2 + y^2 + (z + d)^2)^{3/2}} \tag{4}$$

In the limit $z \to 0$, this gives $E_z(z = 0) = -\frac{2kqd}{(x^2+y^2+d^2)^{3/2}}$. The expression we constructed in Eq. (2) thus satisfies the boundary conditions so it must be the correct, unique, solution.

The expression (2) has a very nice physical interpretation. It is the potential for a real charge, q, at $z = d$ (the first term) and an equal and opposite charge $-q$ at $z = -d$ (the second term). This second "fictitious charge" is called the image charge and provides the inspiration for the image charge method. The solution (2) is only valid for $z > 0$, because for $z < 0$, the solution is $V = 0$. Equation (2) is a solution to Poisson's equation and the superposition method because it is just a sum of the potentials due to two point charges. Now lets calculate some interesting quantities from the solution.

The "screening" or induced charge density at the surface of the conductor is $\sigma(x, y, 0) = \epsilon_0 E_z(x, y, 0)$ which yields,

$$\sigma(x, y, 0) = \frac{-dq}{2\pi(x^2 + y^2 + d^2)^{3/2}} \tag{5}$$

where $k = 1/(4\pi\epsilon_0)$ was used. What is the total induced charge? To find this we need to integrate the surface charge density over the surface, $z = 0$. This is easiest in cylindrical co-ordinates,

$$Q_{induced} = \int_0^{2\pi} \int_0^{\infty} s d\phi ds \frac{-dq}{2\pi(s^2 + d^2)^{3/2}} = -dq \frac{1}{(s^2 + d^2)^{1/2}}|_0^{\infty} = -q \tag{6}$$

We could have figured out that the total induced charge had to be $-q$ by using Gauss's law, so it is a nice check on the calculation.

What is the force on the charge q? For this we can just use the image charge so we have,

$$\vec{F}_q = q\vec{E}_{-q}(x=0, y=0, d) = \frac{-kq^2}{4d^2}\hat{z}. \tag{7}$$

This is a force of attraction so charges are attracted to conductors. For this reason, and other reasons, conductors tend to accumulate excess charge (usually electrons). Finally we would like to know how much work is required to place the charge at position d. Since the force is negative, the work is negative. Note that we cannot use the potential or potential energy of two point charges (q, q'), separated by distance $2d$, for a somewhat subtle reason. The naive calculation would be $W = -\int_{\infty}^{2d}(-kq^2/z^2)dz$, but the correct calculation is,

$$W = -\int_{\infty}^{d} \frac{-kq^2}{4z^2}dz = \frac{-kq^2}{4d} \tag{8}$$

Note that this is one half of the naive value due to the fact that the image charge changes position when the real charge changes position, so the naive calculation does not apply.

Point charge near a grounded conducting sphere

Surprisingly the flat surface calculation discussed above can be extended to the case of a point charge near a grounded conducting sphere and many related problems, for example a line charge near a cylinder. We will go through the simplest case of a point charge near a grounded conducting conductor. Without loss of generality we can place the center of our spherical polar co-ordinate system at the center of the conducting sphere. We then place the real charge, q, at position $\vec{a} = a\hat{z}$. There is azimuthal symmetry so we can restrict consideration to a function $V(r, \theta)$, without any dependence on ϕ. In the spirit of the image charge method we try to find a solution where there is an image charge at position $\vec{b} = b\hat{z}$, where $b < R$ and R is the radius of the conducting sphere.

The voltage boundary condition is $V(R, \theta) = 0$ as the sphere is grounded. As in the flat surface case, we guess the solution for $r > R$ to be a superposition of the potential due to the real charge and due to the image charge.

$$V(r, \theta) = \frac{kq}{|\vec{r} - \vec{a}|} + \frac{kq'}{|\vec{r} - \vec{b}|} \tag{9}$$

Using the cosine rule, this becomes,

$$V(r, \theta) = k[\frac{q}{(r^2 + a^2 - 2racos\theta)^{1/2}} + \frac{q'}{(r^2 + b - 2rbcos\theta)^{1/2}}] \tag{10}$$

The location of and value of the image charge are not known yet, but we can find them by choosing two convenient angles ($\theta = 0, \pi$) and using $r = R$.