PHY481 - Lecture 11: Solutions to Laplace's Equation
Griffiths: Chapter 3

Continuing from Lecture 10, we noted that the solution to a charge placed outside a grounded conducting sphere may be written as,

$$V(r \geq R, \theta) = \frac{kq}{|\vec{r} - \vec{a}|} + \frac{kq'}{|\vec{r} - \vec{b}|} \tag{1}$$

Using the cosine rule, this becomes,

$$V(r, \theta) = k[\frac{q}{(r^2 + a^2 - 2racos\theta)^{1/2}} + \frac{q'}{(r^2 + b^2 - 2rbcos\theta)^{1/2}}] \tag{2}$$

The location of and value of the image charge are not known yet, but we can find them by choosing two convenient angles ($\theta = 0, \pi$) and using $r = R$, so that,

$$\frac{kq}{R-a} + \frac{kq'}{R-b} = 0; \quad \frac{kq}{R+a} + \frac{kq'}{R+b} = 0; \tag{3}$$

Solving these equations yields,

$$q' = \frac{-qR}{a}, \quad b = \frac{R^2}{a} \tag{4}$$

so that,

$$V(r, \theta) = k[\frac{q}{(r^2 + a^2 - 2racos\theta)^{1/2}} - \frac{(qR/a)}{(r^2 + (\frac{R^4}{a})^2 - 2r\frac{R^2}{a}cos\theta)^{1/2}}] \tag{5}$$

First we need to check that $V(R, \theta) = 0$ for all θ, and substituting $r = R$ we have,

$$V(R, \theta) = k[\frac{q}{(R^2 + a^2 - 2Racos\theta)^{1/2}} - \frac{q}{\frac{a}{R}(R^2 + (\frac{R^4}{a})^2 - 2\frac{R^3}{a}cos\theta)^{1/2}}]. \tag{6}$$

It is evident that this is zero for all θ so the boundary condition on the potential is satisfied. Now we need to check the boundary condition on the electric field. The electric field in polar co-ordinates is given by,

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} \tag{7}$$

The $\hat{\phi}$ component is zero as there is no ϕ dependence in the potential. The other two components are,

$$E_r = -kq[-\frac{r - acos\theta}{(r^2 + a^2 - 2racos\theta)^{3/2}} + \frac{(R/a)(r - \frac{R^2}{a}cos\theta)}{(r^2 + (\frac{R^4}{a})^2 - 2r\frac{R^2}{a}cos\theta)^{3/2}}] \tag{8}$$

and

$$E_\theta = -\frac{kq}{r}[\frac{-rsin\theta}{(r^2 + a^2 - 2racos\theta)^{3/2}} + \frac{(R/a)(r\frac{R^2}{a}sin\theta)}{(r^2 + (\frac{R^4}{a})^2 - 2r\frac{R^2}{a}cos\theta)^{3/2}}] \tag{9}$$

Evaluating this expression at the surface of the sphere (i.e. $r = R$) shows that the $E_\theta(r = R) = 0$ is zero as required by the boundary conditions at a conductor surface.

The force on the charge q can either be found directly, i.e. using $kqq'/(b-a)^2$ or by using $\vec{F}_q = q\vec{E}_{q'}$ where $\vec{E}_{q'}$ is the electric field of the image charge, the second term of equations (16) and (17) above, evaluated at the location of the real charge q i.e. at $(r, \theta) = (a, 0)$. We find that $E_\theta(a, 0) = 0$ and $E_r(a, 0)$ gives

$$\vec{E}_{q'} = -kq\frac{(a^3R - aR^3)}{(a^4 + R^4 - 2a^2R^2)^{3/2}}\hat{r} = -kq\frac{aR(a^2 - R^2)}{(a^2 - R^2)^3}\hat{r} \tag{10}$$

Which reduces to

$$\vec{E}_{q'} = \frac{-kqaR}{(a^2-R^2)^2}\hat{r} \quad \text{so that} \quad \vec{F}_q = \frac{-kq^2aR}{(a^2-R^2)^2}\hat{r} \tag{11}$$

The direct expression yields,

$$\vec{F}_q = \frac{kqq'}{(a-b)^2}\hat{r} = \frac{-kq^2aR}{(a^2-R^2)^2}\hat{r} \tag{12}$$

The work required to bring the charge q from infinity to position a is given by,

$$Work = -\int \vec{F}\cdot d\vec{r} = \int_{\infty}^{a} \frac{kq^2zR}{(z^2-R^2)^2}dz' = \frac{kq^2R}{2(a^2-R^2)}. \tag{13}$$

The case of the grounded conducting sphere described above can be generalized to the case of an isolated sphere with total charge Q_0, by placing an appropriate charge at the center of the sphere.

Some other systems solvable using image charge methods

Some problems can be solved using many image charges, one example where four image charges works is the case of a charge q at $(a,a,0)$ when the regions $y<0$ and $x<0$ are conducting and grounded. In that case three image charges $-q$ at $(a,-a,0)$, q at $(-a,-a,0)$ and $-q$ at $(-a,a,0)$ ensure that the potential on the surfaces of the metal regions are all zero. In a similar way, the method of images can be used for a wedge of angle π/n (the case above has wedge angle $\pi/2$) when a charge is placed on the central axis of the wedge. In that case alternating image charges placed at symmetric positions at the centers of all wedges inside the metal sum perfectly to ensure that the potential is zero on the wedge surfaces. One of the problems in Homework 3 is of this type.

The extension to problems where the conductor is at some finite voltage (instead of zero) requires adding charges to produce that voltage. The charges have to be placed symmetrically to ensure that no electric field is generated in the metal. E.g. if we want a sphere of radius R at potential V_0, then we place an image charge Q_0 at the center of the sphere so that $V_0 = kQ_0/R$. This corresponds to distributing the charge Q_0 uniformly on the surface of the sphere. The electrostatic potential for $r>R$ of this problem is found by superposition, i.e. $V(\vec{r}) = V_G(\vec{r}) + kQ_0/r$. In the case of a conducting slab, a sheet of image charge is placed at the center of the slab, while in the case of a conducting cylinder, a line charge is placed at the center of the cylinder.

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In some cases an infinite set of image charges can be used with one example being a charge lying between two grounded metal sheets at locations $z=d$ and $z=-d$. The solution is found by adding image charges iteratively to find,

$$V(r) = kq[\frac{1}{r} + \sum_{k=1}^{\infty} \frac{2(-1)^k}{(r^2+4d^2k^2)^{1/2}} \tag{14}$$

This can be generalized to the non-symmetric case by using a similar procedure. This approach can also be used for some problems involving two spheres or two cylinders.

The methods we used for the flat surface and sphere case above can be used for the case of line charge near a flat surface (use Cartesian co-ordinates), or for a line charge near a conducting cylinder (treat this using cylindrical co-ordinates). In these cases we used a superposition of potentials due to line charges, that are of the form $\lambda ln(r)/(2\pi\epsilon_0) + constant$. The case of a conducting cylinder is given below.

Dipole potential

Consider two point charges in free space; q at position $(0,0,d/2)$ and $-q$ at position $(0,0,-d/2)$. We want to find the electrostatic potential $V(r,\theta,\phi)$ at distance $r >> d$. This is found by doing a leading order expansion in d/r. The exact expression for the dipole potential is (in polar co-ordinates),

$$V(r,\theta) = \frac{kq}{r_+} - \frac{kq}{r_-} = \frac{kq}{(r^2+(d/2)^2 - drcos\theta)^{1/2}} - \frac{kq}{(r^2+(d/2)^2+drcos\theta)^{1/2}} \tag{15}$$

The expansion of this expression at large r corresponds to small d/r, so that,

$$V(r,\theta) = \frac{kq}{r}\left[\frac{1}{(1+(d/2r)^2-(d/r)cos\theta)^{1/2}} - \frac{1}{(1+(d/2r)^2+(d/r)cos\theta)^{1/2}}\right] \approx \frac{kq}{r}\frac{dcos\theta}{r} + O[(d/r)^2] \quad (16)$$

where we used the Taylor expansion, $1/(1+x)^{1/2} = 1 - x/2 + O(x^2)$. The expression on the RHS is the dipole potential and is usually written as,

$$V(r,\theta) = \frac{k(qd)cos\theta}{r^2} = k\frac{\vec{p}\cdot\hat{r}}{r^2} \quad (17)$$

where $\vec{p} = qd\hat{z}$ is the dipole moment (note that its direction is defined from negative to positive charge).