PHY481 - Lecture 12: Solutions to Laplace's equation

Griffiths: Chapter 3

Before going to the general formulation of solutions to Laplace's equations we will go through one more very important problem that can be solved with what we know, namely a conducting sphere (or cylinder) in a uniform field. We will solve this problem using a superposition of solutions including the dipole potential we found above.

Conducting sphere or cylinder in a constant electric field

We consider a grounded conducting sphere placed in a uniform electric field. In this case it is not clear how to use the method of images as there are no charges defined in the problem. The potential due to a uniform electric field $\vec{E} = E_0\hat{z}$ is $-E_0 z$ or in polar co-ordinates $-E_0 r cos\theta$. If we combine this with the point charge and dipole potentials, with arbitrary constants, we have,

$$V(r,\theta) = \frac{A}{r} + B + C\frac{cos\theta}{r^2} + Drcos\theta \quad r > a \tag{1}$$

where a is the radius of the sphere. We know that each term in this expression satisfies Laplace's equation, so the sum does too - that is superposition again. Note that this is not the general form of the solution in spherical co-ordinates, but it will turn out to be sufficient for the problem of a conducting sphere in an electric field. We don't expect the first two terms to contribute as the first corresponds to a net charge on the sphere and the second to a finite potential. By symmetry we expect the first term to be zero while the second term is zero because the sphere is grounded. To ensure that the solution is correct as $r \to \infty$, we need $D = -E_0$. Since the conducting sphere is grounded, we must have the boundary conditions that the potential at the surface of the sphere be equal to zero $V(r = a, \theta) = 0$. If we choose $C = a^3 E_0$, $A = B = 0$ then we satisfy the boundary conditions on the potential so we suspect the solution is,

$$V(r,\theta) = -E_0 cos\theta[r - \frac{a^3}{r^2}] = -E_0 z + k\frac{\vec{p}_{induced}\cdot\hat{r}}{r^2} \tag{2}$$

where the induced dipole moment is given by, $p_{induced} = E_0 a^3/k$. The induced dipole moment is then proportional to the electric field.

The electric field outside the sphere is given by,

$$E_r = -\frac{\partial V}{\partial r} = E_0(1 + \frac{2a^3}{r^3})cos\theta; \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0(1 - \frac{a^3}{r^3})sin\theta \tag{3}$$

Notice that $E_\theta(a,\theta) = 0$ as it must be to ensure that the electric field at the surface of the conductor is perpendicular to the surface. At this point we have satisfied all the boundary conditions so we know the solution is correct and unique.

The induced charge density at the surface of the sphere is calculated from $\sigma(\theta) = \epsilon_0 E_r(r = a) = 3\epsilon_0 E_0 cos\theta$. Integrating over the sphere surface gives the net charge on the sphere,

$$Q = \int_0^\pi 2\pi a^2 sin\theta(3\epsilon_0 E_0 cos\theta)d\theta = 0 \tag{4}$$

which shows that no charge is transferred from ground to the sphere, so this result also applies when the sphere is isolated. Moreover if we integrate over the top hemisphere,

$$Q = \int_0^{\pi/2} 2\pi a^2 sin\theta(3\epsilon_0 E_0 cos\theta)d\theta = 3\pi a^2 \epsilon_0 E_0 \tag{5}$$

we see that a positive charge is induced in the top hemisphere. There is an equal amount of negative charge on the lower hemisphere. Note that the net force on the conducting sphere is zero, as the net force on a dipole in a uniform field is zero. A conducting or dielectric sphere is attracted into an electric field, in order to screen out the field and hence reduce the energy. This is the principle that allows optical tweezers (lasers) to manipulate quantum dots and biomolecules tethered to latex spheres.

If the sphere is charged, with an additional charge Q_0, this additional charge is distributed uniformly on the sphere surface and its effect on the potential is found by superposition.

Exactly the same procedure works for conducting cylindrical in a constant applied electric field, where the solution is,

$$V(r,\phi) = -E_0 cos\phi(s + \frac{a^2}{s}), \tag{6}$$

by following the same reasoning as for the spherical case above. The electric field components are $E_r = E_0 cos\phi(1 + a^2/r^2)$; $E_\theta = -E_0 sin\phi(1 - a^2/r^2)$ and the surface charge density is $2\epsilon_0 E_0 cos\phi$. Integration shows that there is no net charge transfer to the conducting cylinder so this solution also applies for an isolated conducting cylinder in a constant applied field. Addition of a charge to an isolated cylinder leads to a uniform charge density on the surface of the cylinder and an additional term $Alnr$ in the potential.
NOTE: this calculation does not allow the cylinder to move. What if we allow the cylinder to rotate? What orientation do you think it will adopt?

Laplace's equation in the three co-ordinate systems

Now we are ready to look at more general procedures for solving Laplace's equation, $\nabla^2 V = 0$. The Laplacian in any othogonal co-ordinate system is (see Lecture 4),

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u_1}(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3})] \tag{7}$$

For Cartesian co-ordinates we have $u_1 = x, u_2 = y, u_3 = z$, and $h_1 = 1, h_2 = 1, h_3 = 1$, so that,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{8}$$

In cylindrical co-ordinates $u_1 = s, u_2 = \phi, u_3 = z$, and $h_1 = 1, h_2 = s, h_3 = 1$, so that,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}. \tag{9}$$

In spherical polar co-ordinates $u_1 = r, u_2 = \theta, u_3 = \phi$, and $h_1 = 1, h_2 = r, h_3 = rsin\theta$, so that,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}. \tag{10}$$

Problems with dependence on only one co-ordinate (i.e., z or s or r)

In Cartesian co-ordinates, this corresponds to the potential outside a sheet of charge, or equivalently the potential of a constant electric field $-E_0 z$. In fact the general solution is quite trivial, namely,

$$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{so} \quad V(z) = a + bz \tag{11}$$

where a and b are two constants that can be used to fit the boundary conditions. The solution is thus always linear.

In cylindrical co-ordinates, this corresponds to the potential outside a line charge, so we expect a logarithmic solution to be possible (i.e. ln(s)) Laplace's equation in this case is,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) = 0; \quad \text{so} \quad s\frac{\partial V}{\partial s} = b \tag{12}$$

where b is a constant. Integrating we find, $V(s) = a + bln(s)$, which is the general solution in the cylindrical case.

In spherical polar co-ordinates,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) = 0; \quad \text{so} \quad r^2\frac{\partial V}{\partial r} = b \tag{13}$$

Integrating, we find the general solution $V(r) = a + b/r$, which is the potential outside charge distributions with spherical symmetry. The one variable solutions to Laplace's equation are monotonic i.e. decreasing or increasing with no minima or maxima on their interior. Nevertheless electrostatic potential can be non-monotonic if charges are added as a change in slope can occur at charges.

However if a charge is included, the one dimensional solutions to Laplace's equation only apply to the regions of space that do not have charge. They do NOT apply in regions where there is charge, for example inside a sphere with a constant charge density. In that case we have to solve Poisson's equation, for example in the case of a sphere with constant charge density we solve,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) = -\frac{\rho}{\epsilon_0}; \tag{14}$$

We can also solve this. First move the r^2 to the other side and integrate, to find,

$$r^2\frac{\partial V}{\partial r} = -\frac{\rho r^3}{3\epsilon_0} + b \quad \text{Solving yields} \quad V(r) = -\frac{\rho r^2}{6\epsilon_0} - \frac{b}{r} + d \tag{15}$$

Recall that we found the solution in Problem 2:21, $kQ/R + \rho(R^2 - r^2)/(6\epsilon_0)$, which is of course consistent with the solution found from Poisson's equation.

Problems with dependence on two co-ordinates - separation of variables

General properties

First we can easily generalize the remark about monotonicity of solutions to Laplace's equation to higher dimensional cases. The generalization is that if we solve Laplace's equation on a finite domain, there are no maxima or minima of the potential in the interior of the domain. Therefore the minima and maxima only occur on the boundaries.

To see why this is true, consider Lapalace's equation in Cartesian co-ordinates. It is clear that at least one of the terms must be negative and at least one must be positive, implying that in at least one direction the curvature of the solution must be negative and in at least one direction the curvature of the solution must be positive. This implies that there are no minima or maxima which require all positive and all negative curvature respectively. Instead all we can have on the interior of the domain are saddle points. This is very different than solutions in quantum mechanics where the presence of the energy in the equation leads to oscillatory eigenfunctions, where the number of minima and maxima on the domain is related to the energy of the quantum particle.

Cartesian co-ordinates in two dimensions

Laplace's equation is then,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{16}$$

This can no longer be directly integrated, so we try to reduce it to a simpler form by assuming a solution of the form $V(x, y) = X(x)Y(y)$. Subsituting this expression into Eq. (16), we find that,

$$\frac{d^2X}{dx^2}Y + X\frac{d^2Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = constant = -k^2 \tag{17}$$

The requirement that the last expressions have to equal a constant occurs as if we have $f(x) = g(y)$, then both f and g must be constant. This is proven by taking a derivative with respect to x, so that $df/dx = dg/dy = 0$. Then integrate with respect to x to show $f(x) = constant$. It is convenient to call the constant $-k^2$. The sign difference between the X equation and the Y equation, and whether k is real or imaginary, is very important and determines whether the solutions to X or Y are oscillatory or decaying functions. From this expression we have,

$$X(x) = A'(k)e^{ikx} + B'(k)e^{-ikx}; \qquad Y(y) = C'(k)e^{ky} + D'(k)e^{-ky} \tag{18}$$

Often it is convenient form to write the solutions as combinations of odd and even functions, namely,

$$X(x) = A(k)cos(kx) + B(k)sin(kx); \qquad Y(y) = C(k)cosh(ky) + D(k)sinh(ky) \tag{19}$$

The constants A, B, C, D can in general depend on the separation constant k. We can also add the one dimensional solutions ie $ax + b$ and $cy + d$ which correspond to the separation constant $k = 0$. The general solution is then,

$$V(x, y) = (a + bx)(c + dy) + \sum_k [A(k)cos(kx) + B(k)sin(kx)][C(k)cosh(ky) + D(k)sinh(ky)] \tag{20}$$

The values of k that are used depends on the problem and often an infinite series of values is required. That is, we use superposition to add solutions with different values of the separation constant k. This gives us a lot of freedom in choosing which terms to try, and a lot of constants that we can use to find a solution. Usually many of these are removed using symmetry arguments. A similar procedure may be used to find solutions in three dimensions.

Circular co-ordinates

Circular co-ordinates are the (s, ϕ) part of cylindrical polar co-ordinates. The Laplacian in cylindrical polar co-ordinates is given by,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial V}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{21}$$

Assuming that $V = S(s)\Phi(\phi)$, we find,

$$\frac{s}{S}\frac{\partial}{\partial s}(s\frac{\partial S}{\partial s}) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \tag{22}$$

We now separate the equation using,

$$\frac{s}{S}\frac{\partial}{\partial s}(s\frac{\partial S}{\partial r}) = \frac{-1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = k^2 \tag{23}$$

The ϕ equation has solutions $cos(k\phi), sin(k\phi)$ when k is finite. When $k = 0$, it has solution $C + D\phi$. Noting that $\Phi(\phi) = \Phi(\phi + 2\pi)$ is required to ensure that Φ is single valued, we set $k = n$, where $n = 0, 1....$ We could also use n negative, but that would just change the sign of the constants D_n in a general solution of the form $C_n cos(n\phi) + D_n sin(n\phi)$. The s equation has the form,

$$s^2\frac{d^2S}{ds^2} + s\frac{dS}{ds} - n^2S = 0 \tag{24}$$

which for $n \neq 0$ has solutions $S = A_n s^n + B_n s^{-n}$, while for $n = 0$, $S = A + Blns$. With these observations, we find that the general solution to Laplace's equation in circular co-ordinates (ie. cylindrical co-ordinates with no dependence on z) is,

$$V(s,\phi) = (A + Bln(s))(C + D\phi) + \sum_{n=1}^{\infty}(A_n s^n + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi)) \tag{25}$$