PHY481 - Lecture 13: Solutions to Laplace's equation
Griffiths: Chapter 3

Spherical polar co-ordinates

The Laplacian in spherical polar co-ordinates is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{1}$$

We only consider the case where there is no dependence on ϕ, so $V(r,\theta,\phi) \to V(r,\theta) = R(r)\Theta(\theta)$. Note that in Quantum mechanics solutions to the time independent Schrodinger equation with a spherical potential lead to solutions of the form $R(r)Y_l^m(\theta,\phi)$. The solutions we find here are for the special case $m = 0$, so that the Legendre Polynomials we find below are related to the spherical Harmonics through $Y_l^0(\theta,\phi) = a_l P_l(cos\theta)$, where a_l is a constant that is needed due to a different choice of normalization for the spherical harmonics. Assuming no dependence on ϕ and making the subsitution $V(r,\theta) = R(r)\Theta(\theta)$ into Laplace's equation, then dividing through by $R\Theta$ gives,

$$\nabla^2 V = \frac{1}{R}\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) = -\frac{1}{\Theta sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) = C = l(l+1) \tag{2}$$

The reason why we choose the separation constant $C = l(l+1)$ will become clear later. With this choice, we have the two equations,

$$\frac{\partial}{\partial r}(r^2\frac{\partial R}{\partial r}) - l(l+1)R = 0 \tag{3}$$

and

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta}{\partial\theta}) + l(l+1)\Theta = 0 \tag{4}$$

The R equation is solved by $R(r) = A(l)r^l + B(l)/r^{l+1}$. The Θ equation is more interesting and requires a series solution. First we make the substitution, $u = cos\theta$ and $P(u) = \Theta(\theta)$, which imply that

$$\frac{\partial}{\partial\theta} = -sin\theta\frac{\partial}{\partial u} \tag{5}$$

so that,

$$\frac{d}{du}[(1-u^2)\frac{\partial P(u)}{\partial u}] + l(l+1)P(u) = 0 \tag{6}$$

This equation is call Legendre's equation and is solved using a series solution, $P(u) = \sum_n C_n u^n$.

Substituting this series into Legendre's equation we find that,

$$\frac{\partial}{\partial u}\sum_{n=1}^{\infty}(1-u^2)C_n n u^{n-1} + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{7}$$

and,

$$\sum_{n=2}^{\infty} C_n n(n-1)u^{n-2} - \sum_{n=1}^{\infty} C_n n(n+1)u^n + \sum_{n=0}^{\infty} l(l+1)C_n u^n = 0 \tag{8}$$

The coefficient of u^n in this equation must be zero, implying that,

$$C_{n+2}(n+2)(n+1) + C_n[l(l+1) - n(n+1)] = 0; \text{ hence } C_{n+2} = C_n\frac{n(n+1) - l(l+1)}{(n+1)(n+2)} \tag{9}$$

For a given value of l, we get two series of solutions, one starting with a value of C_0 and producing C_2, C_4... and the other starting with C_1 and producing C_3, C_5... The key physical observation is that if l is an integer, the series terminates at $n = l$, leading to a finite polynomial solution. In contrast if l is not an integer, the series does not terminate and the coefficients remain finite at infinity, a solution that does not have physical meaning. The constants

C_0 and C_1 are fixed by requiring that the polynomials be normalized on the interval $[-1, 1]$, which corresponds to $[-\pi, \pi]$ in the original variable θ. The normalization and orthogonality conditions are,

$$\int_{-1}^{1} P_l(u)P_m(u)du = \frac{2}{2l+1}\delta_{ml} \tag{10}$$

The first few Legendre polynomials are,

$$P_0(u) = 1; \quad P_1(u) = u; \quad P_2(u) = (3u^2 - 1)/2; \quad P_3(u) = (5u^3 - 3u)/2 \tag{11}$$

Even l correspond to even functins, while odd l correspond to odd functions. Instead of using Eq. (8) to find the coefficients and Eq. (9) to normalize them, it is more convenient to use a direct recursion formula called Bonnet's formula,

$$(l+1)P_{l+1}(u) = (2l+1)uP_l(u) - lP_{l-1}(u) \quad \textit{with} \quad P_0(u) = 1; P_1(u) = u \tag{12}$$

Setting up Problem 3.22

We need to find the potential inside and outside a sphere of radius R. There is a charge density σ_0 on the upper half of the spherical surface and a charge density on the lower half of the sphere surface. The general solution is,

$$V(r,\theta) = \sum_{l=0} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(cos\theta) \tag{13}$$

For $r > R$ convergence at infinity requires that $A_l = 0$, while for $r < R$ convergence requires that $B_l = 0$, so we have,

$$V(r > R, \theta) = \sum_{l=0} \frac{B_l}{r^{l+1}} P_l(cos\theta); \qquad V(r,\theta) = \sum_{l=0} A_l r^l P_l(cos\theta) \tag{14}$$

Now we need to impose the boundary conditions. First impose the condition that the potential is continuous at $r = R$ (or equivalently that the parallel electric field is continuous), so that

$$\sum_{l=0} \frac{B_l}{R^{l+1}} P_l(cos\theta) = \sum_{l=0} A_l R^l P_l(cos\theta) \quad \text{so that} \quad \frac{B_l}{R^{l+1}} = A_l R^l. \tag{15}$$

Next we need to impose the relation between the perpendicular electric field and the charge density,

$$E_r(r = R^+, \theta) - E_r(r = R^-, \theta) = \frac{\sigma}{\epsilon_0} \tag{16}$$

where $\sigma = \sigma_0$ for $\theta < \pi/2$ and $\sigma = -\sigma_0$ for $\pi/2 < \theta < \pi$. Due to the symmetry of σ we only need to keep odd terms in the sum over l.

— *Aside* — To prove orthogonality, consider Legendre's equation for two polynomials P_m and P_n as follows,

$$P_n[(1-u^2)P_m'' - 2uP_m' + m(m+1)P_m] = 0; \quad P_m[(1-u^2)P_n'' - 2uP_n' + n(n+1)P_n] = 0 \tag{17}$$

where the primes indicate a derivative with respect to u. Subtracting the first from the second of these two equations and using $d/du(P_nP_m' - P_mP_n') = P_nP_m'' - P_mP_n''$ gives,

$$(1-u)^2 \frac{d}{du}(P_nP_m' - P_mP_n') - 2u(P_nP_m' - P_mP_n') + (m(m+1) - n(n+1))P_nP_m = 0 \tag{18}$$

which is equal to,

$$\frac{d}{du}[(1-u^2)(P_nP_m' - P_mP_n')] + (m(m+1) - n(n+1))P_nP_m = 0 \tag{19}$$

Now we integrate this equation over the interval $[-1, 1]$. This integral produces zero for the first term of the equation, so the second must also be zero. Therefore if $m \neq n$ the integral of P_nP_m over this interval must be zero, proving othogonality. — *End of Aside* —

Since $P_l(cos\theta)$ are an orthogonal (and complete) set, we can use them to expand any piecewise continuous function. In spherical co-ordinates, we use them as the basis for a Fourier-type analysis.