PHY481 - Lecture 15: Remarkable general properties of electrostatics
Griffiths: Chapter 3

Earnshaw's theorem

One of the remarkable aspects of Laplace's equation on any domain is that there can be no minima in the interior of the domain, there can only be saddle points. That means the electric field has the property that there is always at least one direction with a positive field and one direction with a negative field. Now consider placing a test charge at any point where the field has been derived from a potential that is a solution to Laplace's equation. Since $\vec{F} = q\vec{E}$, the test charge is not stable as there is always an unstable direction, as is typical of saddle points. This leads to a general and remarkable result of classical electrostatics, Earnshaw's theorem: Charge configurations in electrostatics are unstable, with the opposite charges tending to collapse and like charges moving an infinite distance appart.

Uniqueness theorems

The fact that there are no minima or maxima on the interior regions of a domain also leads in a straightforward way to the uniqueness theorems:

First uniqueness theorem - fixed voltage (Dirichelet) boundary conditions

Lets assume that we have found two solutions to Laplace's equations, V_1 and V_2, where each solution has been found for the same set of fixed voltage, or Dirichelet, boundary conditions. We define $V_3 = V_2 - V_1$. Since the boundary conditions are the same, $V_3 = 0$ on the boundary. Clearly V_3 must also satisfy Laplace's equation but now with boundary conditions zero everywhere. Since solutions to Laplace's equation can have no maxima or minima on the interior of a domain and since both the maxima an minima are zero on the boundary, the only possibility is $V_3 = 0$. Therefore $V_1 = V_2$ and the solutions to Laplace's equation with Dirichelet boundary conditions are unique. The same procedure demonstrates uniqueness for Poisson's equation.

Second uniqueness theorem - fixed total charge on conductors (Neuman b.c.)

This states that if we know the total charge on conductors and Dirichelet boundary conditions on the remaining boundaries then solutions to Laplace's equation (and Poisson's equation) are unique. The proof is a bit longer (see Griffiths p 118).

Green's reciprocity theorem

This is a strange and at first counterintuitive result,

$$\int \rho_1 V_2 d\vec{r} = \int \rho_2 V_1 d\vec{r} \tag{1}$$

It is proven by using Poisson's equation to write,

$$\int \rho_1 V_2 d\vec{r} = -\epsilon_0 \int (\nabla^2 V_1) V_2 d\vec{r} \tag{2}$$

Now use the identity $\nabla \cdot (V_2 \nabla V_1) = V_2 \nabla^2 V_1 + \nabla V_1 \cdot \nabla V_2$. so that,

$$\int \rho_1 V_2 d\vec{r} = \int d\vec{r}[\nabla \cdot (V_2 \nabla V_1) - \nabla V_1 \cdot \nabla V_2] \rightarrow -\int d\vec{r}(\nabla V_1 \cdot \nabla V_2) \tag{3}$$

where the integral $\int \nabla \cdot (V_2 \nabla V_1) d\vec{r}$ is found to be zero by using the divergence theorem and the asymptotic properties of the potential and electric field for a point charge. Now use the identity $\nabla \cdot (V_1 \nabla V_2) = V_1 \nabla^2 V_2 + \nabla V_2 \cdot \nabla V_1$, so that

$$-\int d\vec{r} \nabla V_1 \cdot \nabla V_2) = -\int d\vec{r}[\nabla \cdot (V_1 \nabla V_2) - V_1 \nabla^2 V_2] \rightarrow \int d\vec{r}(V_1 \nabla^2 V_2) \tag{4}$$

and finally

$$\int d\vec{r}(\nabla V_1 \cdot \nabla V_2) = \frac{-1}{\epsilon_0} \int \rho_2 V_1 d\vec{r} \tag{5}$$

prooving the result. This procedure is equivalent to integrating by parts twice to go from the LHS to the RHS and removing all of the excess terms using the divergence theorem and the asymptotic properties of point charges.

An example of series solution in cylindrical co-ordinates

Consider a cylinderical surface of radius R where the top half of the cylindrical surface is at potential V_0 and the bottom half is at potential $-V_0$. Find expressions for the potential on the interior and the exterior of the cylindrical shell.

The general solution in cylindrical co-ordinates, when there is no z-dependence, is given by,

$$V(s,\phi) = (a + b\phi)(c + dln(s)) + \sum_{n=1}^{\infty}(A_n s^2 + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi)) \tag{6}$$

The boundary conditions are odd in ϕ, so we choose $C_n = 0$. We have to construct different solutions on the interior and the exterior. The exterior solution must converge as $s \to \infty$ so we set $A_n \to 0$ and $c = d = 0$, so that,

$$V_{ext}(s,\phi) = \sum_{n=1}^{\infty} \frac{b_n}{s^n} sin(n\phi)) \tag{7}$$

we also need to ensure that the potential is positive for $\phi\varepsilon[0,\pi]$ and negative for $\phi\varepsilon[0,-\pi]$, which is achieved by choosing only odd values of n in the sum above, so that,

$$V_{ext}(s,\phi) = \sum_{n=odd}^{\infty} \frac{b_n}{s^n} sin(n\phi)) \tag{8}$$

Now we impose the boundary condition on the voltage, by evaluating at $s = R$, multiplying both sides by $sin(m\phi)$ and integrating over $[0, 2\pi]$ so that,

$$\int_0^{\pi} V_0 sin(m\phi)d\phi + \int_{\pi}^{2\pi}(-V_0)sin(m\phi)d\phi = \int_0^{2\pi}\sum_{n=odd}^{\infty}\frac{b_n}{R^n}sin(n\phi))sin(m\phi)d\phi = \frac{\pi b_m}{R^m} \tag{9}$$

for m odd. Notice that for m even the LHS is zero confirming our intuitive choice of m odd. Carrying out the integrals gives,

$$\frac{-2V_0}{m}[cos(m\pi) - cos(0)] = \frac{4V_0}{m} = \frac{\pi b_m}{R^m}; \quad \text{or} \quad b_n = \frac{4V_0 R^n}{\pi n} \tag{10}$$

Therefore,

$$V_{ext}(s,\phi) = \sum_{n=odd}^{\infty} \frac{4V_0}{\pi n}\frac{R^n}{s^n} sin(n\phi) \tag{11}$$

The calculation for the interior solution is the same except that we take $B_n = 0$ to find,

$$V_{int}(s,\phi) = \sum_{n=odd}^{\infty} \frac{4V_0}{\pi n}\frac{s^n}{R^n} sin(n\phi) \tag{12}$$

A. A dipole in an electric field

The simple view: In the simplest case, we consider a simple dipole consisting of two charges, placed in a uniform electric field. We take the angle between the electric field and the dipole to be θ. Since $\vec{F} = q\vec{E}$ there are equal and opposite forces on the two charges of magnitude q in the dipole, so the net force is zero and the net center of mass motion is zero. However there is a torque on the dipole

$$\vec{N} = \vec{r} \wedge \vec{F} = 2\frac{d}{2}sin\theta qE = \vec{p} \wedge \vec{E} \tag{13}$$

where d is the separation between the two charges in the dipole, and $\vec{p}$ is the dipole moment. The torque is zero when the dipole aligns with the field and $\theta = 0$. The state of zero energy is taken to be at the angle $\theta = 90^o$ where the torque is maximum. The potential energy of the dipole in the field is then found from,

$$U = (-)\int_{\pi/2}^{\theta} pEsin\theta'(-)d\theta' = -pEcos\theta = -\vec{p}\cdot\vec{E} \tag{14}$$

The lowest energy state is when $\theta = 0$ and the dipole is aligned with the applied field. The highest energy state is when $\theta = \pi$ and at this point the torque is also zero, so it is a point of unstable equilibrium. Any slight change in θ away from π makes the dipole have a torque pushing it to towards the lowest energy state.

General calculation: For a general charge distribution, we have,

$$\vec{N} = \int \vec{r} \wedge d\vec{F} = \int \vec{r} \wedge \rho \vec{E} d\vec{r} = \int \rho \vec{r} \wedge \vec{E} d\vec{r} \tag{15}$$

If the field is uniform, then the electric field can be taken out of the integral. We also use the general expression for the dipole $\vec{p} = \int \rho(\vec{r})\vec{r}d\vec{r}$ to find,

$$\vec{N} = \vec{p} \wedge \vec{E} \tag{16}$$

The energy of a dipole in a field can be found by starting with the general expression for the energy cost of placing a small amount of charge a fixed potential,

$$U = \int \rho(\vec{r}')V(\vec{r}')d\vec{r}' = \int \rho(\vec{r}')[V(\vec{r}) + (\vec{r}' - \vec{r}) \cdot \vec{\nabla} V + ...]d\vec{r}' \tag{17}$$

Using $E = -\nabla V$, assuming that the electric field is constant, and taking the total charge $\int \rho(\vec{r}')d\vec{r}' = 0$ to be zero, we find that the last expression on the RHS reduces to,

$$\int \rho(\vec{r}')\vec{r}'d\vec{r}' \cdot \vec{\nabla} V = -\vec{p} \cdot \vec{E} \tag{18}$$

If the total charge is not zero, or if the electric field is not constant, then the dipole will have a center of mass motion in addition to it's rotation toward alignment with the field.