PHY481 - Lecture 16: Force between wires, Ampere's law, $\vec{F} = i\vec{l} \wedge \vec{B}$
Griffiths: Chapter 5

Force between charged wires and between current carrying wires

Magnetostatics is the study of static magnetic fields and the direct currents (DC) that generate them. The MKS unit of magnetic field is the Tesla (T) and often the magnetic field is given in Gauss (G), the CGS unit. The relation is $10,000G = 1T$, so small fields are often given in Gauss.

In the early 1800's several researchers noted that there is a force between two parallel wires that carry a steady current . If the wires are separated by distance d they observed that,

$$\frac{|F|}{l} = \frac{\mu_0}{2\pi d} i_1 i_2 \quad \text{attractive for currents in the same direction} \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7} N/A^2$ is the permeability of vacuum. If the currents are in opposite directions the wires repel. This result can be compared to the force between two infinite parallel line charges, with charge densities λ_1, λ_2. The magnitude of the force between the line charges, when separated by d, is given by,

$$\frac{|F|}{l} = \frac{q_2}{l}|E_{\lambda_1}| = \frac{\lambda_1 \lambda_2}{2\pi \epsilon_0 d} \quad \text{repulsive for like charges} \tag{2}$$

where we used the electric field near a line charge $E(s) = \lambda/2\pi\epsilon_0 s$. Other than the direction of the force these two results are similar. Clearly, excess charge leads to electric fields and a direct current leads to magnetic fields. The total force is a superposition of these two effects, so that electrostatics and magnetostatics don't affect each other.

Also in the 1820's Ampere and Oersted also noticed that magnets are affected by a DC current in a wire and Ampere found that a wire carring a DC current is surrounded by a magnetic field of the form,

$$\vec{B}(s) = \frac{\mu_0 i}{2\pi s}\hat{\phi} \quad \text{Magnetic field near a wire} \tag{3}$$

This should be compared to the electric field of a line charge

$$\vec{E}(s) = \frac{\lambda}{2\pi \epsilon_0 s}\hat{s} \quad \text{Electric field near a line charge} \tag{4}$$

The most remarkable difference between these two results is the *direction* of the fields
- The electric field diverges from the line charge and is curl free ($\vec{\nabla} \wedge \vec{E} = 0$)
- The magnetic field forms circles around the steady current and is divergence free ($\vec{\nabla} \cdot \vec{B} = 0$) Notice that the magnetic field is not moving even though the steady current is flowing and even though we often talk about the circulation of the magnetic field about the current-The magnetic field is static. The direction of circulation is given by the right hand rule, applied to either the magnetic field or to the current. Ampere thus demonstrated that steady state currents are the *sources* of static magnetic fields. This enables the enormously important technologies that use electromagnets in motors, generators, MRI's etc. The technological importance of these discoveries cannot be overstated.

Ampere's law and problems solved using it

Ampere realised that his measurements for the magnetic field near a wire may be written in the form of a path integral.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i = \mu_0 \int \vec{j} \cdot d\vec{a} \quad Ampere's\ law \tag{5}$$

Here $d\vec{l}$ is a small vector along the direction of the path. For example for a circle the unit vector is tangent to the circle. Now consider the magnetic field around a long straight wire, where the current, i, is in the $\hat{k}$ direction, then we have,

$$\int_0^{2\pi} B(s)\hat{\phi} \cdot sd\phi\hat{\phi} = 2\pi sB(s) = \mu_0 i \tag{6}$$

Solving for $B(s)$ yields,

$$\vec{B}(s) = \frac{\mu_0 i}{2\pi s}\hat{\phi} \quad \text{outside wire} \tag{7}$$

Ampere's law is correct only for DC currents. Later in the course we will add an additional term which enables us to use this equation even when there are time dependent fields - the Maxwell displacement current term.

Ampere's law problem 1: A wire with circular cross-section and radius b, carrying current density j_0
First recall that the relation between current and current density,

$$I = \int \vec{j} \cdot d\vec{a} \tag{8}$$

For a wire of radius b and constant current density j_0, the current enclosed at radius s is then,

$$i_{enclosed} = \pi b^2 j_0 \quad s > b; \quad \text{and} \quad i_{enclosed}(s) = \pi s^2 j_0 \quad s < b. \tag{9}$$

We take a circular contour and take the direction of the current in the wire to be $\hat{z}$. Then the RHR shows that the magnetic field is in the $\hat{\phi}$ direction. The contour integral is then the same as before, i.e. $\oint \vec{B} \cdot d\vec{l} = 2\pi s B(s)$. Since we have the enclosed current in the equation above, we can simply solve for $\vec{B}$ in the two regimes by using the different enclosed current in the two regimes, so that,

$$\vec{B}(s) = \frac{\mu_0 b^2 j_0}{2s} \hat{\phi} \quad s > b; \quad \text{and} \quad \frac{\mu_0 s j_0}{2} \hat{\phi} \quad s < b \tag{10}$$

Ampere's law problem 2: A uniform sheet of current, with current per unit length $K\hat{x}$
Assume that the sheet lies in the x-y plane and the the current is in the positive $\hat{x}$ direction. By the right hand rule (RHR), we find that the magnetic field is in the negative $\hat{y}$ direction above the sheet, while it is in the positive $\hat{y}$ direction below the sheet. We therefore choose a rectangular contour, with the long sides above and below the sheet and parallel to the $\hat{y}$ direction. By symmetry there is no field component in the $\hat{z}$ direction, so the parts of the contour in that direction give no contribution. The magnitude of the magnetic field is also the same above and below, but in opposite directions. We can then evaluate the path integral to find that,

$$2LB = \mu_0 L K; \quad \text{so that} \quad \vec{B} = -\frac{1}{2}\mu_0 K\hat{y}, \ z > 0; \quad \vec{B} = \frac{1}{2}\mu_0 K\hat{y}, \ z < 0. \tag{11}$$

Ampere's law problem 3: A long thin solenoid with n turns per unit length and radius R
We place the axis of the solenoid on the z-axis and define the current to be circulating in the counterclockwise direction. We take the solenoid to be infinite in length, with radius R with its axis along the z-axis, where there are n "turns" per unit length. In that case, if we draw a rectangular contour around the whole solenoid, it is evident that the field outside the solenoid is, by symmetry, zero. Using the RHR, if the solenoid current is rotating in the $\hat{\phi}$ direction (counterclockwise), the magnetic field is along the $\hat{z}$ direction. The finite field inside the solenoid is found from,

$$LB = \mu_0 L n i; \quad \text{so that} \quad B = \mu_0 n i \quad r < 0 \tag{12}$$

where i is the current in the circuit. Large magnetic fields can be achieved by increasing n, or by increasing i, which is the basic goal in design of electromagnets. Copper electromagnets can achieve magnetic fields of up to $10T$, though values around $1 - 5T$ are more standard. Superconducting electromagnets can achieve higher values, but require liquid Helium cooling. Large magnets used in physics experiments, such as the NSCL, LHC etc, use superconducting electromagnets. Use of magnets in NMR and in medicine (MRI) also require large magnetic fields and rely on electromagnets. Electromagnets provide very nice control of the size of the magnetic field. William Stugeon invented the electromagnet in 1823. High temperature superconductors hold the promise of superconducting magnets operating at liquid Nitrogen instead of Liquid He which would be a very important advance.

The force on a current carrying wire in a magnetic field, $\vec{F} = i\vec{l} \wedge \vec{B}$
In electrostatics the force on a charge q_2 that is placed in an electric field generated by charge q_1 is $\vec{F}_2 = q_2\vec{E}_1$. In fact that is how we calculated the force between two lines of charge above. What is the analogous result for the two current carrying wires? We would like to find an expression for the force that is similar to this. When we treated the lines of charge the force is of the form $\lambda_2 l\vec{E}_1$. The analogous form for the magnetic field is $i_2 l\vec{B}_1$ however this is in

the wrong direction! A little thought show that the expression $i_2\vec{l}\wedge\vec{B}_1$ is in the direction found in the experiments. We therefore postulate (as Ampere did) that the force on a currrent carrying wire in a magnetic field is given by,

$$\vec{F} = i\vec{l}\wedge\vec{B} \tag{13}$$

where, as in the electrostatics case, the magnetic field does NOT include the field generated by i itself. For the case of two current carrying wires, we have,

$$\vec{F} = i_2\vec{l}_2\wedge\frac{\mu_0 i_1\hat{\phi}}{2\pi d} \tag{14}$$

It is clear that this is the same as Eq. (1) above. To be specific, choose two parallel wires carrying current in the $\hat{z}$ direction and separated by distance d. Take wire 1 to be at the origin and wire 2 to be at position $d\hat{y}$. The magnetic field due to wire 1 at the location of wire 2 is then $\mu_0 i_1\hat{\phi}/2\pi d$.