PHY481 - Lecture 18: Biot-Savart law, magnetic dipoles, vector potential
Griffiths: Chapter 5

Biot-Savart law for infinite wire

Ampere's law is convenient for cases with high symmetry, but we need a different approach for cases where the current carrying wire is not so symmetric, for example in current loops. The Biot-Savart law is equivalent to Ampere's law and is a superposition method for magnetostatics,

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\vec{l} \wedge \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{id\vec{l} \wedge \vec{r}}{r^3}; \quad \text{Biot} - \text{Savart law} \tag{1}$$

We found the magnetic field around a long straight wire using Ampere's law $\vec{B} = \mu_0 i\hat{\phi}/(2\pi s)$, where the current i is in the $\hat{z}$ direction. Now let's use the Biot-Savart law to find this result. We again use cylindrical co-ordinates and note that by symmetry only the $\hat{\phi}$ component is finite, so we only consider,

$$d\vec{B}_\phi(s,z) = \frac{\mu_0 i}{4\pi}\frac{dz\hat{z} \wedge \vec{r}}{r^3} = \frac{\mu_0 i}{4\pi}\frac{dz sin(\alpha)}{r^2} \tag{2}$$

where α is the angle between $\hat{z}$ and $\vec{r}$ and $r^2 = s^2 + z^2$ and $sin(\alpha) = s/r$. The magnetic field at position s is then

$$\vec{B}(s) = \frac{\mu_0 i}{4\pi}\int_{-\infty}^{\infty}\frac{s\ dz}{(s^2+z^2)^{3/2}}\hat{\phi} \tag{3}$$

The integral,

$$\int_{-\infty}^{\infty}\frac{s\ dz}{(s^2+z^2)^{3/2}} = s\frac{z}{s^2(s^2+z^2)^{1/2}}|_{-\infty}^{\infty} = \frac{2}{s} \tag{4}$$

so that,

$$\vec{B}(s) = \frac{\mu_0 i}{2\pi s}\hat{\phi} \tag{5}$$

which of courses agrees with the result found using Ampere's law.

Other cases where the Biot-Savart law is integrated to find the magnetic field include, circular and rectangular loops, discs and spheres. A common problem is to take a disc or sphere with a constant charge density and to spin it at a constant rate. This leads to currents that generate a static magnetic field.

Magnetic field of a magnetic dipole

There are two sources of magnetic dipoles: currents and intrinsics magnet moments of elementary particles. In either case, we have a magnetic dipole moment $\vec{m}$ that gives the magnitude and direction of the magnetic dipole. This magnetic dipole moment is directly analagous to the electric dipole moment $\vec{p}$, so we can write,

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}; \quad \vec{\tau} = \vec{m}\cdot\vec{B}; \quad U = -\vec{m}\cdot\vec{B} \tag{6}$$

In the next section we argue that the magnetic moment of a current ring is given by,

$$\vec{m} = i\vec{a}, \tag{7}$$

where $\vec{a}$ is the area of the current loop and the direction is normal to the current loop. To make this comparison, we note that if we choose $\vec{m} = m\hat{z}$ in Eq. (6), then the field on the $z - axis$ is

$$\vec{B} = B_z\hat{z} = \frac{\mu_0}{4\pi}\frac{2m}{z^3} \quad \text{on z} - \text{axis} \tag{8}$$

A circular current ring

Consider a circular loop of radius s centered at the origin and lying in the x-y plane. A steady current, i, flows in the positive $\hat{\phi}$ direction. Find the magnetic field on the z- axis. We use the Biot-Savart where $d\vec{l} = sd\phi\hat{\phi}$ and $\hat{r}$ is along the vector from a position on the circle to a point on the z-axis. If we take the angle to the z axis to be α, from the geometry we find that $d\vec{l} = sd\phi\hat{\phi}$ is perpendicular to $\hat{r}$. The vector $d\vec{l} \wedge \hat{r}$ makes an angle of $90 - \alpha$ to the z-axis and its projection onto the x-y plane is at angle ϕ to the x-axis. On the z-axis, by symmetry, the magnetic field points in the z-direction, and we find,

$$B_z(z) = \frac{\mu_0 i}{4\pi}\int_0^{2\pi} \frac{sin(\alpha) s d\phi}{s^2+z^2} = \frac{\mu_0 i}{2}\frac{s^2}{(s^2+z^2)^{3/2}} \tag{9}$$

Expanding the expression above at large distances, z, gives,

$$B_z(z) \approx \frac{\mu_0 i s^2}{2z^3}(1-\frac{3}{2}\frac{s^2}{z^2}) \approx \frac{\mu_0 i a}{2\pi z^3} \tag{10}$$

Comparing with Eq. (8) we see that this is the same as for a dipole provided we set $m = ia$.

General relations for the magnetic field

From Ampere's law, we have,

$$\oint \vec{B}\cdot d\vec{l} = \int (\vec{\nabla}\wedge\vec{B})\cdot d\vec{a} = \mu_0 i = \mu_0 \int \vec{j}\cdot d\vec{a} \tag{11}$$

leading to the differential form of Ampere's law,

$$\vec{\nabla}\wedge\vec{B} = \mu_0\vec{j} \tag{12}$$

Since there are no magnetic monopoles we also have,

$$\oint \vec{B}\cdot d\vec{a} = \int(\vec{\nabla}\cdot\vec{B})d\vec{r} = 0 \quad \text{so that} \;\; \vec{\nabla}\cdot\vec{B} = 0 \tag{13}$$

We can also write the general form of the Biot-Savart law for a current density $\vec{j}$,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{d\vec{r}'\vec{j}(\vec{r}')\wedge(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}; \quad \mathrm{Biot-Savart\ law} \tag{14}$$

Taking the divergence of this equation confirms that $\vec{\nabla}\cdot\vec{B} = 0$.

The vector potential

In magnetostatics the magnetic field is divergence free, and we have the vector identity $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{F}) = 0$ for any vector function $\vec{F}$, therefore if we write $\vec{B} = \vec{\nabla}\wedge\vec{A}$, then we ensure that the magnetic field is divergence free. $\vec{A}$ is the vector potential, and despite being a vector simplifies the calculations in some cases. It is also very important in quantum mechanics where the solution to Schrodinger's equation in a magnetic field involves adding a term proportional to $\vec{A}$ to the momentum operator. To find a differential equation for the vector potential, we use the differential form of Ampere's law to find,

$$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \mu_0\vec{j} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \tag{15}$$

where we used a vector identity to write the last expression on the RHS. This still looks messy, however now note that the choice of $\vec{A}$ is not unique as we can write $\vec{A}+\vec{\nabla}f$ and $\vec{\nabla}\wedge\vec{A}$ is unaltered. We can then choose the scalar function f to help solve problems. In electrostatics a convenient choice is $\vec{\nabla}\cdot\vec{A} = 0$ as this removes the first term in the last expression on the RHS of Eq. (15). This choice is called the Coulomb gauge, and in this gauge we have,

$$\nabla^2\vec{A} = -\mu_0\vec{j}; \qquad \text{Coulomb gauge.} \tag{16}$$

This is just Poisson's equation for each of the components of $\vec{A}$. Recall that for the voltage $\vec{\nabla}^2 V = -\rho/\epsilon_0$. To show that it is always possible to find a scalar function f to ensure that $\vec{\nabla}\cdot\vec{A} = 0$, consider a problem where we have found a solution $\vec{A}_s$ where $\vec{\nabla}\cdot\vec{A}_s \neq 0$. We then need to find a scalar function f_s so that $\vec{\nabla}\cdot(\vec{A}_s+\vec{\nabla}f_s) = 0$. This reduces to $\nabla^2 f_s = -\vec{\nabla}\cdot\vec{A}_s$. This is just Poisson's equation again. It always has a solution, so we can always find a scalar f_s so that $\vec{A} = \vec{A}_s+\vec{\nabla}f_s$ is divergence free.

Now that we have found that the vector potential in the Coulomb gauge obeys Poisson's equation, the solution to these equations in integral form may be written down,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|}; \quad \text{while in electrostatics} \qquad V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}')d^3r'}{|\vec{r}-\vec{r}'|}. \tag{17}$$