PHY481 - Lecture 2: Vector calculus

Griffiths: Chapter 1 (Pages 10-38)

The Gradient - A vector derivative operator

In Cartesian co-ordinates the change in a scalar function is,

$$df = dx\frac{\partial f}{\partial x} + dy\frac{\partial f}{\partial y} + dz\frac{\partial f}{\partial z} \tag{1}$$

If we define the gradient as the vector derivative operator which in Cartesian co-ordinates is defined through,

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \tag{2}$$

Then the gradient of a scalar funtion (f) is,

$$\vec{\nabla} f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) \tag{3}$$

More importantly, a co-ordinate system independent representation of df may now be written,

$$df = f(\vec{l} + d\vec{l}) - f(\vec{l}) = d\vec{l} \cdot \vec{\nabla} f. \tag{4}$$

This expression contains a powerful result that we use all the time in electrostatics where the electric field is related to the electrostatic potential through $\vec{E} = -\vec{\nabla} V$. To see this result, consider surfaces on which the scalar function, f (or V in electrostatics), is a constant. In electrostatics these surfaces are equipotentials. In gravitational systems they are surfaces of constant gravitational potential or at the earth surface they are contours of constant elevation. If we consider a displacement $d\vec{l}$ along an equipotential (or contour of constant f), then clearly $df = 0$. Since the LHS of Eq. (4) is zero, then the right hand side (RHS) of this equation must also be zero. Since $d\vec{l}$ is finite the only way for the RHS to be zero is for $\vec{\nabla} f$ to be perpendicular to the surface of constant f, since then $d\vec{l}$ is perpendicular to $\vec{\nabla} f$ so their dot product is zero. In electrostatics this is the same as the statement that the electric field lines are perpendicular to equipotential surfaces. For example consider the electrostatic potential of a sheet of charge with charge density σ and lying in the x-y plane. Then $V(z) = -\frac{\sigma z}{2\epsilon_0}$. The electic field is then $\frac{\sigma}{2\epsilon}\hat{z}$.

The Divergence

The divergence is the dot product of the gradient operator and a vector function, $\vec{F} = (F_x, F_y, F_z)$, so that

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{5}$$

Gauss's law in differential form is

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \tag{6}$$

Note that the density and electric field are both at position $\vec{r}$. If there is no charge at that position, then the divergence of the electric field at that point should be zero. First consider the electric field of a point charge $\vec{E} = kQ\vec{r}/r^3$. The divergence of this quantity is,

$$\vec{\nabla} \cdot (\frac{kQ\vec{r}}{r^3}) = kQ[\frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3})] \tag{7}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$. Carrying out the derivatives gives,

$$\frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} = 0 \tag{8}$$

This is zero everywhere except at $r = 0$ where it is singular. The charge is at position $r = 0$ and to treat this we need the concept of a delta function which we will discuss later. The main point is that at positions where there is no charge, the divergence of $\vec{E}$ is zero. Note that this is a special case of the general picture of what divergence looks like. In the case of the electric field the particular function $\vec{r}/r^3$ happens to be zero even though the form of the field lines look like the divergence should be finite. If we consider a generalized vector function, such as $\vec{F} = \vec{r}/r^n$, then we find that,

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^n} = \frac{1}{r^n} - \frac{nx^2}{r^{n+2}} + \frac{1}{r^3} - \frac{ny^2}{r^5} + \frac{1}{r^3} - \frac{nz^2}{r^5} = \frac{(3-n)}{r^n} \tag{9}$$

According to the differential form of Maxwell's equation an electic field of this sort can only be generated by a finite charge density at position $\vec{r}$, $\rho(\vec{r}) = kQ\epsilon_0 \frac{3-n}{r^n}$. It is easy to construct vector functions that have zero divergence, for example consider the magnetic field near a straight wire with current $I\hat{z}$, found using Ampere's law and cylindrical co-ordinates,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad \text{so that,} \quad 2\pi B(s) = \mu_0 I \quad \text{or} \quad \vec{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\phi} \tag{10}$$

In Cartesian co-ordinates $s^2 = x^2 + y^2$ and $\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y} = \frac{-y}{s}\hat{x} + \frac{x}{s}\hat{y}$, so that $\vec{B} = \frac{\mu_0 I}{2\pi s^2}(-y, x, 0)$. We can now take the divergence in Cartesian co-ordinates,

$$\frac{\mu_0 I}{2\pi}\left(\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right) = 0 \tag{11}$$

The divergence of magnetic fields found using Ampere's law are zero. Of course that agrees with Gauss's law for magnetic fields $\vec{\nabla} \cdot \vec{B} = 0$ which could be finite only if there were magnetic monopoles. Another function to consider is simply $\vec{F} = z\hat{z}$. This function has finite divergence. Can you think of a charge distribution that could generate an electric field of this form? Note: A function that has zero divergence is called solenoidal. Gauss's law for magnetism shows that magnetic fields are always solenoidal, while in electostatics electric fields are solenoidal only in regions of space where there is no net electric charge.

The Curl

The curl of a vector function, $\vec{\nabla} \wedge \vec{F}$ is defined in the same way as the cross product

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \tag{12}$$

Lets look at this function for the two cases above, i.e. the electric field of a point charge and the magnetic field of a wire, where $\vec{B} = \frac{\mu_0 I}{2\pi s^2}(-y, x, 0)$. First the case of wire where we find that,

$$\vec{\nabla} \wedge \vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}, \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}, \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{z} = \left(\frac{1}{s^2} - \frac{2x^2}{s^4} + \frac{1}{s^2} - \frac{2y^2}{s^4}\right)\hat{z} = 0 \tag{13}$$

This is zero away from the wire as there are no sources away from the wire. If the magnetic field had been described by a function like, $(-y, x, 0)/s^n$ with $n \neq 2$, then the curl would have been non-zero and Ampere's law would state how much current density must be present. Now consider one component of the curl the electric field due to a point charge $\vec{r}/r^3$, for example the x-component,

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = \frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right) = 0 \tag{14}$$

It is easy to show that this is always true, even for functions like $\vec{r}/r^n$. In general Faraday's law shows that any electric field in electrostatics has zero curl. Functions that have zero curl are called irrotational. In electrostatics electric fields are irrotational and magnetic fields are irrotational only in regions of space where there are no current sources.

Product and quotient rules

Six different product rules for the expressions $\vec{\nabla}(fg)$, $\vec{\nabla}(\vec{A} \cdot \vec{B})$, $\vec{\nabla} \cdot (f\vec{A})$, $\vec{\nabla} \cdot (\vec{A} \wedge \vec{B})$, $\vec{\nabla} \wedge (f\vec{A})$, $\vec{\nabla} \wedge (\vec{A} \wedge \vec{B})$ are given in Griffiths. They are derived using elementary expansions and product rules. There are also three quotient rules for the expressions $\vec{\nabla}(f/g)$, $\vec{\nabla} \cdot (\vec{A}/g)$, $\vec{\nabla} \wedge (\vec{A}/g)$. One of the assigned problems is to prove these quotient rules.

Vector triple products - combinations of div, grad, curl

Four important ways to combine div grad and curl are as follows.

1. The simplest is the divergence of the gradient of a scalar function, $\vec{\nabla}\cdot(\vec{\nabla}f) = \nabla^2 f$. The operator ∇^2 is one of the most important in mathematical physics and is called the Laplacian. e.g. Poisson equation (EM), Schrodinger equation (QM), wave equation, diffusion equation etc all contain Laplacians.

2. We may also take the curl of the gradient of a scalar function and this turns out to be zero, $\vec{\nabla}\wedge(\vec{\nabla}f) = 0$. This is easily proved using the determinant form of the cross product.

3. A second combination that leads to zero is the divergence of the curl of a vector function, $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{F}) = 0$. This is easily proved using the determinant form of the vector triple product $\vec{A}\cdot(\vec{B}\wedge\vec{C})$.

4. Finally we can form the curl of a curl and this can be reduced using the vector triple product identity on cross products, $\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}$. This is easily proved using the cross product identity $\vec{A}\wedge(\vec{B}\wedge\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$, with $\vec{A} = \vec{B} = \vec{\nabla}$.

Integration of vector functions

We are familiar with integration of scalar functions in one, two or three dimensions and these skills are important in EM. In addition two types of integration of vector functions are important: Line integrals and surface integrals. In particular the integral forms of Maxwell's equations are written in terms of these integrals. The fundamental theorem of calculus for a scalar function states that,

$$\int_a^b \frac{\partial f}{\partial x} dx = \int_a^b df = f(b) - f(a) \tag{15}$$

Consider a line integral form of this expression,

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = \int_{\vec{a}}^{\vec{b}} df = f(\vec{b}) - f(\vec{a}) \tag{16}$$

which is the fundamental theorem for gradients. Note that the path integral of a gradient of a scalar function only depends on the values of the function at the endpoints - it is then a "state function".

The fundamental theorem for the divergence (Gauss's theorem, Green's theorem or **the divergence theorem**) is

$$\int_\nu (\vec{\nabla}\cdot\vec{v}) d\tau = \oint \vec{v}\cdot d\vec{a} \tag{17}$$

This can be viewed as a conservation law. The LHS gives the sources within the volume ν and the RHS gives the total flow through any closed surface enclosing the sources.

The fundamental theorem for the curl (**Stokes theorem**) is given by,

$$\int_S (\vec{\nabla}\wedge\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l} \tag{18}$$

Notice that if we are given a contour P, then any surface consistent with that contour can be used. Also if the area $\vec{a}$ forms a closed surface then the RHS is zero. Since the curl gives the circulation, the integral of circulation over a surface leads to lots of cancellation of flows in opposite direction. The result is only the circulation or flow at the surface remains.

Maxwell's equations: From integrals to derivatives

We now have all the tools we need to go from the integral forms of Maxwell's equations to the derivative forms. Lets start with Gauss's law,

$$\oint \vec{E}\cdot d\vec{a} = \int_\nu \vec{\nabla}\cdot\vec{E} d\tau = q/\epsilon_0 = \frac{1}{\epsilon_0}\int_\nu \rho d\tau \tag{19}$$

where we used the divergence theorem and the fact that the charge q is the volume integral of the charge density ρ. It is clear that Gauss's law is satisfied if the arguments of the volume integrals on the LHS and RHS are set equal, ie. $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, which is the differential form of Gauss's law. One may ask whether the solutions to the differential and integral forms are always the same, as we seem to be removing some information by removing the integral. It turns out that for almost all situations the solutions are the same. A similar procedure applies to the second of Maxwell's equations $\oint \vec{B}\cdot d\vec{a} = 0$, leading to $\vec{\nabla}\cdot\vec{B} = 0$.

Now consider Faraday's law

$$\oint \vec{E}\cdot d\vec{l} = \int (\vec{\nabla}\wedge\vec{E})\cdot d\vec{a} = \frac{-\partial\phi_B}{\partial t} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a} \tag{20}$$

where we used Stokes theorem and the definition of the magnetic flux. This equation is clearly satisfied if the arguments of the area integrals on the LHS and the RHS are set equal to each other giving, $\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, which is Faraday's law in differential form. Again we can ask if the differential form has the same solutions as the integral form, with the same answer as above - in all the important aspects, yes. The same procedure is used for the Ampere-Maxwell equation where we also need the relation between current and current density, $I = \int \vec{j} \cdot d\vec{a}$. Then we find the differential form of the Ampere-Maxwell law, $\vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$