PHY481 - Lecture 20: Calculating the magnetic vector and scalar potentials
Griffiths: Chapter 5

Calculating the vector potential

Recall that for the solenoid, we found that

$$A(s<R) = \frac{\mu_0 nis}{2}; \quad A(s>R) = \frac{\mu_0 niR^2}{2s} \tag{1}$$

Outside the solenoid there is no magnetic field, but the potential is non trivial. Lets check that $\vec{\nabla}\wedge\vec{A} = 0$ for $s > R$. Since the only component of $\vec{A}$ that is finite is A_ϕ, so that using the expression for the curl in cylindrical co-ordinates, we find,

$$\vec{\nabla}\wedge\vec{A} = [\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}]\hat{s} + [\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}]\hat{\phi} + \frac{1}{s}[\frac{\partial(sA_\phi)}{\partial s} - \frac{\partial A_s}{\partial\phi}]\hat{z} \tag{2}$$

Current sheet The third case that we looked at was an infinite sheet with current density $\vec{K} = K\hat{x}$ lying in the x-y plane. What is the vector potential in that case? Using Ampere's law, we find that the magnetic field for $z > 0$ is $\vec{B}(z>0) = -\mu_0 K\hat{y}/2$. Taking a rectangular contour with normal $-haty$, we find,

$$[A(z) - A(a)]L = \frac{\mu_0 KL}{2}(a-z) \tag{3}$$

so that $\vec{A}(z) = -\frac{\mu_0 Kz}{2}\hat{x}$. The gauge degrees of freedom have been removed from this expression.

The scalar magnetic potential

In many problems we can use a scalar magnetic potential that is analogous in many ways to the electrostatic potential, however it does not have the same basic significance as the electrostatic potential or the vector potential. The scalar magnetic potential can be used in regions of space where there are no currents, so that $\vec{nabla}\wedge\vec{B} = 0$. In that case we can introduce a scalar potential ϕ so that $\vec{B} = -\vec{\nabla}\psi$. We also have $\vec{\nabla}\cdot\vec{B} = 0$ so that $\nabla^2\psi = 0$. Note that the boundary conditions that we use on ψ are the magnetic field conditions. The scalar magnetic potential simplifies many problems, for example 5.11 of Griffiths, where we want to find the vector potential and magnetic field of a spherical shell spinning with angular velocity $\vec{\omega}$. The shell has charge per unit area σ and radius R. First we find the current per unit length $\vec{K} = \sigma R\omega sin\theta\hat{\phi}$, which follows from problem 5.6 of homework 4. Since the scalar magnetic potential must satisfy Laplace's equation, we use the solution in spherical polars,

$$\psi^{out} = \sum_l \frac{B_l}{r^{l+1}}P_l(cos\theta); \quad \psi^{in} = \sum_l A_l r^l P_l(cos\theta) \tag{4}$$

with the boundary conditions

$$B_{\parallel}^{out} - B_{\parallel}^{in} = \mu_0 K; \quad B_{\perp}^{out} - B_{\perp}^{in} = 0 \tag{5}$$

Since $\vec{B} = -\vec{\nabla}\psi$, using the gradient in spherical polars we find,

$$-\frac{\partial\psi^{out}}{\partial r}|_R + \frac{\partial\psi^{in}}{\partial r}|_R = 0 \quad \text{so that} \quad \frac{(l+1)B_l}{R^{l+2}} + lA_lR^{l-1} = 0 \tag{6}$$

Now we note that K is proportional to $sin\theta$ so we only need the $l = 1$ term in the Legendre series, so that

$$-\frac{1}{R}\frac{\partial\psi^{out}}{\partial\theta}|_R + \frac{1}{R}\frac{\partial\psi^{in}}{\partial\theta}|_R = \mu_0\sigma R\omega sin\theta; \quad \text{gives} \quad \frac{B_1}{R^3}sin\theta - A_1 sin\theta = \mu_0\sigma R\omega sin\theta \tag{7}$$

In that case Eq. (6) is $2B_1/R^3 + A_1 = 0$. Using this in Eq. (7) we find that $A_1 = -2\mu_0\sigma R\omega/3$, $B_1 = \mu_0\sigma R^4\omega/3$. The scalar potential is then,

$$\psi^{out} = \frac{\mu_0\sigma R^4\omega}{3r^2}cos\theta; \quad \psi^{in} = \frac{-2\mu_0\sigma R\omega}{3}rcos\theta \tag{8}$$

Using $\vec{B} = -\vec{\nabla}\psi$ in spherical polars yields,

$$\vec{B}^{in} = \frac{2}{3}\mu_0\sigma R\omega(cos\theta\hat{r} - sin\theta\hat{\theta}) = \frac{2}{3}\mu_0\sigma R\omega\hat{z} \quad \vec{B}^{out} = \frac{\mu_0\sigma R^4\omega}{3r^3}(2cos\theta\hat{r} + sin\theta\hat{\theta}) \tag{9}$$

Finally the vector potential may be found by noting that $\vec{B} = \vec{\nabla} \wedge \vec{A}$ in this problem implies that,

$$\frac{1}{rsin\theta}[\frac{\partial}{\partial\theta}(sin\theta A_{\phi}^{in})] = \frac{2}{3}\mu_0\sigma\omega R\omega cos\theta \tag{10}$$

Solving yields $A_{\phi}^{in} = \frac{2}{3}\mu_0\sigma\omega R\omega r sin\theta$. This is a relatively straigtforward way to solve this problem.

Vector potential of a magnetic dipole produced by a current loop

By analogy with the electrostatic case, we deduced that when the magnetic dipole is $\vec{m} = m\vec{z}$

$$\vec{B}_{dipole} = \frac{\mu_0}{4\pi r^3}[2mcos\theta\hat{r} + msin\theta\hat{\theta}] \tag{11}$$

We want to find a vector potential so that $\vec{B}_{dipole} = \vec{\nabla} \wedge \vec{A}$. The curl in spherical polars is,

$$\vec{\nabla} \wedge \vec{A} = \frac{1}{rsin\theta}[\frac{\partial}{\partial\theta}(sin\theta A_{\phi}) - \frac{\partial A_{\theta}}{\partial\theta}]\hat{r} + \frac{1}{r}[\frac{1}{sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_{\phi}]\hat{\theta} + \frac{1}{r}[\frac{\partial}{\partial r}(rA_{\theta}) - \frac{\partial A_r}{\partial\theta}]\hat{\phi} \tag{12}$$

The $\hat{r}$ and $\hat{\theta}$ terms must be finite while the $\hat{\phi}$ term must be zero. We also know that a current ring produces a magnetic dipole. If the current ring is in the x-y plane, then the current is in the $p\hat{h}i$ direction, therefore the magnetic potential is also in the $\hat{\phi}$ direction. For the $\hat{r}$ component we then have,

$$\frac{1}{rsin\theta}[\frac{\partial}{\partial\theta}(sin\theta A_{\phi})] = \frac{\mu_0 mcos\theta}{2\pi r^3} \tag{13}$$

Solving yields

$$A_{\phi} = \frac{\mu_0}{4\pi}\frac{msin\theta}{r^2} \tag{14}$$

The vector form is then,

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{msin\theta}{r^2}\hat{\phi} \quad \text{or} \quad \vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\wedge\hat{r}}{r^2} \tag{15}$$

where the last expression does not rely on choosing $\vec{m}$ to lie in the $\hat{z}$ direction. Using the curl in spherical polars it is easy to show that this reproduces the magnetic field of a dipole

There is a generalization of the expression to cases where a current loop does not lie in the plane. In that case, the magnetic moment is generalized to,

$$\vec{m} = i\int d\vec{a} \tag{16}$$

For example if we have a loop consisting of a rectangle with sides l, a and normal $\hat{x}$, connected to a rectangle with sides l, b and normal $\hat{y}$, then the moment is $\vec{m} = il(a\hat{x} + b\hat{y}$. Proof of the general form is somewhat involved and is given in Griffiths and reproduced below.

Multipole expansion of the vector potential of a current loop

The vector potential of a current ring is given by,

$$\vec{A} = \frac{\mu_0}{4\pi}\int id\vec{l'}\frac{1}{|\vec{r} - \vec{r'}|} = \frac{\mu_0 i}{4\pi}\int d\vec{l'}\sum_l \frac{(r')^l}{r^{l+1}}P_l(cos(\theta') \tag{17}$$

To proceed further we need to introduce a couple of new definitions. First the vector area of a surface, S, is defined to be,

$$\vec{a} = \int d\vec{a} \tag{18}$$

With this there is a nice identity,

$$\oint(\vec{c}\cdot\vec{r})d\vec{l} = \vec{a}\wedge\vec{c} \tag{19}$$

To prove this consider a special case of Stokes theorem with $\vec{v} = \vec{c}T$ where $\vec{c}$ is a constant and T is a function. i.e.

$$\oint (\vec{c}T) \cdot d\vec{l} = \oint [\vec{\nabla} \wedge (\vec{c}T)] \cdot d\vec{a} = -\int [\vec{c} \wedge \vec{\nabla}T] \cdot d\vec{a} = -\vec{c} \cdot \int \vec{\nabla}T \wedge d\vec{a} \tag{20}$$

we therefore have,

$$\oint T d\vec{l} = \int \vec{\nabla}T \wedge d\vec{a} \tag{21}$$

Now use $T = \vec{c} \cdot \vec{r}$. Since $\vec{\nabla}(\vec{c} \cdot \vec{r}) = \vec{c}$, we have,

$$\oint (\vec{c} \cdot \vec{r}) d\vec{l} = \int d\vec{a} \wedge \vec{c} \tag{22}$$

The first two terms of the multipole expansion are,

$$\vec{A} = \frac{\mu_0 i}{4\pi} \oint d\vec{l'} [\frac{1}{r} + \frac{r'}{r^2} cos(\theta') + ...] \tag{23}$$

The first term is zero due to integration of the $d\vec{l'}$ over a closed loop. The second term is,

$$\vec{A}_{dipole} = \frac{\mu_0 i}{4\pi r^2} \oint d\vec{l'} r' cos(\theta') = \frac{\mu_0 i}{4\pi r^2} \oint d\vec{l'} (\hat{r} \cdot \vec{r}') \tag{24}$$

Using $\vec{c} = \hat{r}$ in the above yields $\oint d\vec{l'}(\hat{r} \cdot \vec{r}') = -\hat{r} \wedge \int d\vec{a}' = -\hat{r} \wedge \vec{a}$. The vector potential for a dipole is then,

$$\vec{A}_{dipole} = \frac{\mu_0 \vec{m} \wedge \hat{r}}{4\pi r^2} \rightarrow \frac{\mu_0}{4\pi} \frac{m sin\theta}{r^2} \hat{\phi} \tag{25}$$

where $\vec{m} = i\vec{a}$ and the last expression is for the case where the dipole moment lies along the $\hat{z}$ direction.