

PHY481 - Lecture 21: Faraday's law - motional emf
Griffiths: Chapter 7

emf is called the electromotive force, but it is a voltage

There are many forces that can produce motion of the positive (“holes”) and negative (“electrons”) carriers in conductors and semiconductors. All of these forces are lumped together into the term “electromotive force”. Unfortunately the e.m.f that is used in real calculations, though produced by a force, does not have the units of force. emf is actually an integral of the electric field, so it has the units of voltage. A local electric field or force (“ $q\vec{E}$ ”) produces the motion of the carriers. In some cases, the emf (“ \mathcal{E} ”) is produced in one part of a circuit, for example in the case of a battery or a solar cell. In other cases the emf is distributed around a loop. This is the case of Faraday’s law.

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} \quad (1)$$

In words, Faraday’s law states that a change in magnetic flux leads to an induced emf in any loop surrounding that changing flux. Faraday did many experiments to validate this law, two general types are

- (i) Motional emf where the magnetic field is constant, but the area through which the flux passes is changed.
- (ii) Cases where the area is constant, but the magnetic field changes with time.

Note that there does NOT have to be a wire loops for this emf to occur, it occurs for example in propagating electromagnetic waves, as we shall see later. We shall first look at cases where there is a moving rod or rotating loop - so called motional emf.

Also note that if a coil is placed in a changing magnetic field the Faraday emf is induced. If two coils are placed one on top of each other, then each coil has the Faraday emf induced in it. In motors and generators a large number of coils, N , is used to produce high voltage. In transformers, two different values of N are used to change the voltage.

A. Faraday’s law and motional emf

Rod moving on conducting rails

Consider a uniform and constant magnetic field, B , directed along the z-axis. Now consider moving a conducting rod, of length l , which is directed along the y-direction at constant speed v along the x-direction. The rod is connected to conducting rails. The flux in the loop is changing so that Faraday’s law tells us that there is an induced emf. If the length of the rod is l and the length of the other side of the rectangular loop is L , then we have $L = vt$, provided we assume that the conducting rod starts at the origin at time $t = 0$. The rate of change of the flux is given by,

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -Bl\frac{dL}{dt} = -Blv \quad (2)$$

This emf is induced around the loop.

Recall that in a resistive circuit $V = IR$ and $P = VI = I^2R$. The current in the loop is thus,

$$i = \frac{Blv}{R} \quad (3)$$

where R is the resistance of the loop (rod plus rails). The direction of the current is to oppose the changing flux. Since the flux is increasing, the current flows clockwise in the x-y plane. This produces an induced flux in the z-direction which opposes the changing flux induced by the motion of the rod through the uniform B field. Note that the induced current is small if the resistance of the loop is large, while the induced current is large if the resistance is small.

In cases where an induced current flows, eg. a conducting loop, there is dissipation in the loop. This energy loss must be equal to the work done by an external force, but what is the origin of the force - it is the Lorentz force and the most convenient form in this problem is $d\vec{F} = id\vec{l} \wedge \vec{B}$. In the simple example above, the rod direction is perpendicular to the field, so we find that the magnitude of the magnetic drag force is,

$$F_{drag} = i l B = \frac{B^2 l^2 v}{R} \quad \text{direction opposite } \vec{v} \quad (4)$$

In general a moving magnet near a conductor leads to induced “eddy” currents that oppose the motion. This effect can be strong leading to strong braking effects, e.g. when a magnet is dropped down a copper tube. The external force provides power given by,

$$P_{external} = \vec{F}_{external} \cdot \vec{v} = i L B v = \frac{B^2 L^2 v^2}{R} = \frac{\mathcal{E}^2}{R} = P_{dissipated} \quad (5)$$

The mechanical power that is supplied is dissipated as resistive losses due to current flow in the loop. Surprisingly, the smaller the resistance the larger the resistive losses and the greater the drag force.

Rotating loops - Electric Generators and motors

We consider for illustration the case of a homopolar generator/motor (1831) where a coil is rotated through a constant magnetic field. This generator/motor has the disadvantage that the coils rotate, requiring the use of commutators, as in the DC motor. Tesla (1880's) introduced the idea of rotating the magnet inside the coils which removes the need for the commutator. Modern AC motors and generators are almost exclusively induction devices having stationary coils and rotating magnets. The principle of operation of these devices is basically the same physics as magnetic braking or drag as described above. However in the case of generators, the current generated is shunted to carry out work in the electrical grid.

Consider a coil with N turns and area A rotating about its central axis at constant angular speed ω . The angle between the magnetic moment of the loop and the applied magnetic field is ψ . This angle increases as $\psi(t) = \omega t$ due to the constant angular speed of the coil. The rate of change of the magnetic flux is

$$\frac{d\phi_B}{dt} = \frac{d}{dt}(BA\cos(\psi)) = -BA\omega \sin(\omega t) \quad (6)$$

The induced emf is given by Faraday's which gives,

$$\mathcal{E} = -N \frac{d\phi_B}{dt} = NBA\omega \sin(\omega t) \quad (7)$$

The potential energy is given by,

$$U = -\vec{m} \cdot \vec{B} = NIAB \cos(\omega t) \quad (8)$$

The torque is given by,

$$\tau = \vec{m} \wedge \vec{B} = NIAB \sin(\omega t) \quad (9)$$

The mechanical power that must be supplied to rotate the coil is,

$$P_{mech} = \vec{\tau} \cdot \vec{\omega} = NIAB\omega \sin(\psi) = \frac{N\mathcal{E}AB\omega \sin(\psi)}{R} = \frac{(NAB\omega \sin(\omega t))^2}{R} \quad (10)$$

This must be equal to the electrical power delivered to the load, this is given by,

$$P_{elec} = \frac{\mathcal{E}^2}{R} = \frac{(NBA\omega)^2}{R} \sin^2(\omega t) \quad (11)$$

The average power dissipated is

$$P_{av} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(NBA\omega)^2}{R} \sin^2(\theta) d\theta \quad (12)$$

Using the result that,

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\theta) d\theta = \frac{1}{2} \quad (13)$$

we have,

$$P_{av} = \frac{1}{2} \frac{(NBA\omega)^2}{R} \quad (14)$$

which is the average power delivered by our generator. This is the number you see quoted on your lightbulb.

Moving rods - no conducting loops

Now consider cases where an isolated rod is moving through a constant magnetic field. We can now form a rectangular loop in the same way that we did for the case of the rod moving on rails, except now we take the limit where the conductivity of the rails goes to zero. What happens? The answer is that charges move to set up a voltage

across the rod that balances the motional emf, so the electric field inside the rod is $E = -emf/l$, where $emf = -Blv$ as before. Therefore $E = Bv$, so the electric force balances the magnetic force. This again implies that the Lorentz force law and Faraday's law have the same physical content.

Another popular example is to consider a conducting rod that is pinned at one end and rotating with angular speed ω . In that case, the area of the loop is $A = \phi R^2/2$, where ϕ is the angle. The induced emf is then $emf = -BR^2\omega/2$.

An important deduction from these examples is that when an observer moves through a constant magnetic field, the observer sees an electric field. The effect of transformation to a moving co-ordinate system is then very interesting and is believed to be the way in which Einstein first started thinking about relativity.

No rods at all

Now consider a case where there are no rods, or equivalently, take the limit as the conductivity of the rod goes to zero. In that case the charges do not move, but we still have motional $emf = -Blv$. What does this mean?