PHY481 - Lecture 22: Current, current density, resistive materials
Griffiths: Chapter 7

The continuity equation

The current and current density are related through,

$$i = \int \vec{j} \cdot d\vec{a} = \frac{dQ}{dt} \tag{1}$$

If we write,

$$Q = \int \rho(\vec{r}) d\tau \quad \text{so that} \quad i_{out} = -\frac{dQ}{dt} = \oint \vec{j} \cdot d\vec{a}, \tag{2}$$

then,

$$-\int \frac{\partial \rho}{\partial t} d\tau = \int d\tau \vec{\nabla} \cdot \vec{j} \tag{3}$$

leading to the continuity equation,

$$-\frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{j}. \tag{4}$$

The negative sign is due to the fact that if i_{out} is positive, then dQ/dt is negative.

Note that the continuity equation does NOT include sources. If there are sources of current then it is generalized to,

$$\frac{\partial \rho_s}{\partial t} - \frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{j}. \tag{5}$$

where s labels the source of charge. Note that Gauss's law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ is similar, with the charge itself acting as a source of electric field lines.

From local conduction to resistance - linear isotropic response

We briefly consider resistance at three levels: Macro (R); local continuum (σ_r); atomic (τ). These days connecting these scales is called multiscale modeling, but clearly it has been around for a long time. Physics is usually at the atomic scale while engineering used to be at the continuum and macro scale. These days however everyone is supposed to understand all of these levels.

Everyone is familiar with Ohm's law, $V = IR$, but what does this mean in terms of Maxwell's equations? To make this connection, we define a local version of Ohm's law,

$$\vec{j} = \sigma_r \vec{E} \tag{6}$$

where $\sigma_r = 1/\rho_r$ is the conductivity and ρ_r is the resistivity. This is an experimentally observed relation that we will discuss more below. First lets see the consequences of this "constitutive law" on Maxwell's equations, particularly in the case where there is no net charge on a wire and the magnetic field is time independent, then if we assume that σ_r is uniform and constant the continuity equation and Maxwell's equations imply that for steady (DC) currents in regions with no sources,

$$\vec{\nabla} \cdot \vec{E} = 0; \quad \vec{\nabla} \wedge \vec{E} = 0 \quad \text{so that} \quad \nabla^2 V = 0. \tag{7}$$

That is, the voltage obeys Laplace's equation and $\vec{E} = -\vec{\nabla} V$. This allows us to connect σ_r with R. Here are two examples.

Rectangular slab

Consider a slab of conductivity σ_r, thickness d and area A with its bottom surface at $z = 0$. Find the relation between the conductivity and the resistance of the slab. *Solution:* Apply a voltage V_0 to the top of the slab and ground the bottom. The solution to Laplace's equation is just $V(z) = a + bz$ in this one dimensional problem. Using $V(0) = 0; V(d) = V_0$, we find $V(z) = V_0 z/d$. The current density is I/A and the electric field is V_0/d, so that $\vec{j} = \sigma_r \vec{E}$ becomes, $I/A = \sigma_r V/d$, so that $R = d/(\sigma_r A)$.

Concentric cylinders

Find the resistance between two concentric metal cylindrical shells where the inner shell has radius a and is held at voltage V_0, while the outer shell has radius b and is held at voltage 0. The region between the two metal shells contains a material with conductivity σ_r. The voltage between the two shells is given by $V(s) = c_1 + c_2 ln(s)$, with $V(a) = V_0$, and $V(b) = 0$, so that

$$V_0 = c_1 + c_2 ln(a);\ 0 = c_1 + c_2 ln(b) \quad \text{so that} \quad c_2 = \frac{V_0}{ln(a/b)},\ c_1 = -\frac{V_0 ln(b)}{ln(a/b)} \ \text{ and } \ V(s) = \frac{V_0 ln(s/b)}{ln(a/b)} \tag{8}$$

The electric field and current are then,

$$j(s) = \frac{I}{2\pi sL}; \qquad E = -\frac{\partial V}{\partial s} = \frac{V_0}{ln(b/a)}\frac{1}{s} \tag{9}$$

We then have,

$$j = \sigma_r E \ \text{ implies } \ \frac{I}{2\pi sL} = \sigma_r \frac{V_0}{ln(b/a)}\frac{1}{s} \quad \text{or} \quad R = \frac{ln(b/a)}{2\pi L\sigma_r} \tag{10}$$