PHY481 - Lecture 25: Electromagnetic waves
Griffiths: Section 9.2.1, 9.2.2

Travelling waves and the wave equation

Any function of the form $f(k(x-vt))$ describes a traveling wave. This is simply due to the fact that if we choose a value of $s = x - vt$, then $x = s + vt$ increases at a constant rate, with the rate equal to the velocity of the wave.

Harmonic or linear waves travelling along the x-direction are described by the functions,

$$f(x,t) = Acos(kx - \omega t + \delta); \quad f(x,t) = Bsin(kx - \omega t + \delta); \quad f(x,t) = Ce^{i(kx-\omega t+\delta)} \tag{1}$$

where $k = 2\pi/\lambda$; $\omega = 2\pi f$; $v = f\lambda$. δ is a phase factor that can often be set to zero. These functions are solutions to the wave equation in one dimension,

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{2}$$

The also solve the wave equation in three dimensions.

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{3}$$

However the three dimensional equation has more general solutions of the form,

$$y(x,t) = Acos(\vec{k}\cdot\vec{r} - \omega t + \delta); \quad y(x,t) = Bsin(\vec{k}\cdot\vec{r} - \omega t + \delta); \quad Ce^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)} \tag{4}$$

$\vec{k}$ defines the direction of propagation of the wave, so for example if the wave is propagating along the $\hat{x}$ direction, then $\vec{k} = k_x\hat{x}$. For a general propagation direction $\omega = v|k|$. These expressions also provide solutions to any component of a vector function obeying the wave equation,

$$\nabla^2 \vec{F} = \frac{1}{v^2}\frac{\partial^2 \vec{F}}{\partial t^2}. \tag{5}$$

For example a solution for $\vec{F}$ that is linearly polarized in the $\hat{z}$ direction would be

$$F_x = 0, F_y = 0, F_z = Ce^{i(\vec{k}\cdot\vec{r}-\omega t+\delta)} \tag{6}$$

The direction of propagation is $\hat{k}$, while the direction of polarization is $\hat{n}$. More general solutions that are combinations of the linearly polarized solutions are possible, for example circularly polarized light.

Electromagnetic waves

Optics, Electricity and Magnetism were considered to be unrelated subjects prior to 1820. In 1820 Ampere and Oersted demonstrated that electric current influences magnets and founded the field of magnetostatics. Faraday extended this to include the effect of time varying magnetic fields. However, it was not until 1864 that optics and other electromagnetic waves were unified and their relation to electricity and magnetism was made clear, by James Clerk Maxwell, though addition of his displacement current term, and then solving the equations to show that they predict wave motion that describes EM waves across all frequencies. This prediction was subsequently confirmed by Hertz.

The demonstration of EM waves is actually quite straightforward. In free space, there are no wires so the term $\mu_0\vec{j}$ is not needed, and there are no charges so we remove the term ρ/ϵ_0 from Gauss's law, so that,

$$\vec{\nabla}\cdot\vec{E} = 0; \quad \vec{\nabla}\cdot\vec{B} = 0; \quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}; \quad \vec{\nabla}\wedge\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}; \qquad \text{(in free space)} \tag{7}$$

If we take a time derivative of Ampere's law and use Faraday's law, we find,

$$\nabla\wedge(\nabla\wedge\vec{E}) = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \tag{8}$$

If we take a time derivative of Faraday's law then use Ampere's law, we find,

$$\nabla\wedge(\nabla\wedge\vec{B}) = -\frac{\partial^2\vec{B}}{\partial t^2} \tag{9}$$

An identity that is easy to prove (e.g. using Mathematica!) is,

$$\nabla \wedge (\nabla \wedge \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F} \tag{10}$$

Now note that in free space $\nabla \cdot \vec{E} = \nabla \cdot \vec{B} = 0$, Using these expressions in Eqs. (8) and (9), we find,

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \tag{11}$$

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \tag{12}$$

These are both wave equations, which just means that they have solutions which are of the form,

$$E_x(x, y, z) = E_0 cos(kz - \omega t + \delta) \quad ; \quad E_y = 0 \quad ; \quad E_z = 0 \tag{13}$$

The sin function also works, and solutions like this apply to the y direction and to the z direction. We have to choose the solutions to fit the equations and the initial conditions. Eq. (18) describes an EM wave that travels in the $z - direction$ and whose electric field oscillates in the x direction. Similar solutions exist for waves travelling in the $x - direction$ and in the $y - direction$. We have freedom to choose the direction of motion and also the direction of polarization, as well as the phase in each case - there is thus a lot of freedom.

It is essential to first understand one component of this general solution, so lets consider a function E_x, traveling in the z-direction, as given above. We take $E_y = E_z = 0$ so this is a linearly polarized wave that is polarized in the x direction. If we subsitute this expression into Eq.(11), we find,

$$-k^2 = -\mu_0 \epsilon_0 \omega^2 \tag{14}$$

or, using $k = 2\pi/\lambda$, $\omega = 2\pi f$, $c = f\lambda$,

$$\frac{1}{\mu_0 \epsilon_0} = (f\lambda)^2 = c^2 \tag{15}$$

This demonstrates that the velocity of the wave is,

$$c = \frac{1}{(\mu_0 \epsilon_0)^{1/2}} = 3.0 \times 10^8 m/s \tag{16}$$

This discovery is considered one of the most important of the 19th century and provides the unification of optics, electricity and magnetism. Since $E_y = E_z = 0$, we have,

$$(\nabla \wedge \vec{B})_x = \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t} = \omega E_0 \mu_0 \epsilon_0 sin(kz - \omega t + \delta) \tag{17}$$

We also have,

$$(\nabla \wedge \vec{B})_x = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \rightarrow \frac{-\partial B_y}{\partial z} = \omega E_0 \mu_0 \epsilon_0 sin(kz - \omega t + \delta) \tag{18}$$

If we integrate this expression with respect to z, we find,

$$B_y = \frac{E_0}{c} cos(kz - \omega t + \delta) = \frac{E_x}{c} \tag{19}$$

This expression shows that the magnetic field is oscillating in phase with the electric field.

The electric field oscillates in the x direction, the magnetic field oscillates in the y-direction and the wave travels in the z-direction. From Maxwell's equations in free space it is evident that $\vec{E}$, $\vec{B}$ and the direction of motion are mutually perpendicular. These are the basic properties of EM waves and the full solution is found by superposition. The direction of motion is usually denoted by $\hat{k}$ and we have $\hat{B} = \hat{k} \wedge \hat{E}/c$.

The directions we chose above are not special and we can write the cosine and sine parts of the solution in a unified form. We also define $\vec{k}$ as the vector direction of motion and write for a monochromatic, linearly polarized EM wave,

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}; \quad \vec{B}(\vec{r}, t) = \frac{1}{c} \hat{k} \wedge \vec{E}. \tag{20}$$

If the direction of polarization is $\hat{n}$ and the direction of propagation is $\hat{k}$, then the direction of the magnetic field is $\hat{k} \wedge \hat{n}$.