PHY481 - Lecture 28: Dielectric materials - problem solving
Griffiths: Chapter 4

Boundary value problems

The boundary conditions on the displacement field and electric field across a surface that has surface free charge density σ_f and surface bound charge density σ_b are (using arguments like those used in Lecture 9),

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f \quad \text{so that} \quad \epsilon_{above}E^{\perp}_{above} - \epsilon_{below}E^{\perp}_{below} = \sigma_f \tag{1}$$

$$\vec{E}^{||}_{above} - \vec{E}^{||}_{below} = 0; \quad E^{\perp}_{above} - E^{\perp}_{below} = (\sigma_f + \sigma_b)/\epsilon_0 \tag{2}$$

The electrostatic potential is continuous across the surface.

A uniform electric field applied to a uniform dielectric sphere - Use a similar method for Problem 4.22

We consider an uncharged uniform dielectric sphere of radius a and dielectric constant ϵ, in a constant applied field

$$\vec{E}_0 = E_0\hat{z}, \quad \text{so that} \;\; V = -E_0 z = -E_0 r cos\theta. \tag{3}$$

Since the dielectric sphere is uniform and there is no free charge, we have,

$$\vec{\nabla}\cdot\vec{D} = 0 = \vec{\nabla}\cdot(\epsilon\vec{E}) = \epsilon\vec{\nabla}\cdot\vec{E} = 0. \quad uniform\ \epsilon \tag{4}$$

Using $\vec{E} = -\vec{\nabla}V$ we then find that V still obeys Laplace's equation, so we try the solutions,

$$V_{int} = -C_1 r cos\theta, \quad V_{ext} = -E_0 r cos\theta + \frac{C_2 a^3 cos\theta}{r^2} \tag{5}$$

We impose continuity of V, and the condition $\epsilon_0 E^{ext}_n(a,\theta) = \epsilon E^{int}_n(a,\theta)$, to find,

$$-E_0 + C_2 = -C_1; \quad -E_0 - 2C_2 = -\frac{\epsilon}{\epsilon_0}C_1 \tag{6}$$

which lead to,

$$C_1 = E_0\frac{3}{\epsilon_r + 2}; \quad C_2 = E_0\frac{\epsilon_r - 1}{\epsilon_r + 2} \tag{7}$$

where $\epsilon_r = \epsilon/\epsilon_0$. Using the fact that $V_{int} = -C_1 z$, we find that the electric field inside the sphere is uniform

$$\vec{E}_{int} = -\frac{\partial V_{int}}{\partial z}\hat{z} = E_0\frac{3}{\epsilon_r + 2}\hat{z} \tag{8}$$

and that the dipole of the sphere is

$$\vec{p}_{sphere} = \frac{C_2 a^3}{k}\hat{z} = 4\pi\epsilon_0 E_0 a^3\frac{\epsilon_r - 1}{\epsilon_r + 2}\hat{z} \tag{9}$$

The polarization density is then,

$$\vec{P} = \frac{\vec{p}_{sphere}}{volume} = \frac{\epsilon_0 E_0}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}\hat{z} \tag{10}$$

Again it is useful to compare this to the limit of a metal where $\epsilon_r \to \infty$.

Other homework hints

Problem 4.26 Use the formula for the energy density, $u = \epsilon E^2/2$, and integrate.

Problem 4.28: The energy gain when the dielectric oil enters the cylindrical cavity is $[C_{after} - C_{before}]V^2/2$. Since V is fixed, we need the expression for the capacitance of a cylindrical capacitor when part of the cylindrical cavity has dielectric susceptibility χ_e. The capacitance of a cylindrical capacitor uniformly filled with material of dielectric constant ϵ is $C = 2\pi L\epsilon/ln(b/a)$, where L is the length of the cylinder. When the oil fills the cavity, the capacitance is like two capacitors in parallel. The energy gain when the oil fills the cavity is then $(C(h) - C(0))V^2/2$. The total energy change when the oil enters the cylindrical cavity to height h is then $U = \delta U_{grav} - \delta U_{cap}$. The equilibrium condition is $-\vec{\nabla} U = 0$ which sets the force to zero. In this problem the gradient reduces to a derivative with respect to h.

Problem 4.32 Straightforward applicaton of formulae.

Problem 4.34 The potential has a dipole form outside the sphere and has the form of a dipole plus a constant electric field (along the dipole axis) inside the sphere. (Give arguments as to why this is true) We then take,

$$V_{in} = (a_1 r + \frac{p}{4\pi\epsilon r^2}) cos\theta; \qquad V_{out} = \frac{b_1}{r^2} cos\theta \tag{11}$$

and use the potential and displacement field boundary conditions to find a_1 and $b - 1$.

Problem 4.40 Much of this is done in Lecture 26 of the online notes.