PHY481 - Lecture 29: Magnetic materials
Griffiths: Chapter 6

Magnetic materials

Extension of Maxwell's equations to treat magnetic fields inside and outside magnetic materials is achieved in manner that in some ways is like the treatment of the dielectric response of materials - through analysis of dipoles, in the magnetic case of course we consider magnetic dipoles. To treat a large number of aligned electric dipoles inside a material using Maxwell's equations, we introduced the polarization $\vec{P}$, the electric dipole density. To treat a large number of aligned magnetic dipoles using Maxwell's equations we introduce an analogous quantity, the magnetization $\vec{M}$ which is the magnetic moment density. In the dielectric case, atoms and molecules have either an induced or intrinsic electric dipole moment. In a similar way, atoms and molecules may have an intrinsic magnetic moment and/or a magnetic moment induced by an applied magnetic field.

Atomic origins of magnetic properties

The magnetic properties of atoms comes from a combination of the orbital motion of the electrons about the nucleus and the intrinsic magnetic moment of electrons, protons and neutrons. The orbital response produces a magnetic moment that opposes an applied magnetic field, while the spin part typically produces a magnetic moment that enhances an applied magnetic field.

The fundamental quantum of electron magnetic moment - the Bohr magneton

If a single electron moves in a circular orbit at speed v and with radius r, then the current is

$$I = \frac{e}{2\pi r} v \tag{1}$$

The magnetic moment, $m_{orbital}$ is then,

$$m_{orbital} = I\pi r^2 = -\frac{1}{2} evr = -\frac{e}{2m_e} L \tag{2}$$

where $L = m_e vr$ is the angular momentum of the electron. This relation is usually written as,

$$\vec{m}_{orbital} = g_L \vec{L} \tag{3}$$

where $g_L = -\frac{e}{2m_e}$ is called the gyromagnetic ratio.

According to Bohr theory, the angular momentum is quantized, so that,

$$L = l\hbar \tag{4}$$

where $l = 0, 1, 2...$ is an integer. We then have,

$$m_{orbital} = -\frac{e}{2m_e}\hbar l = -m_B l \tag{5}$$

where we have defined the Bohr magneton,

$$m_B = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24} Am^2 \tag{6}$$

In addition, the electron carries an intrinsic magnetic moment due to its spin. The intrinsic magnetic moment of the electron has a value very close to m_B.

The total magnetic moment is found by a vector addition of the orbital and spin contributions. This sum is called $\vec{m}_0$. The magnetization used in Maxwell's equations are the density of aligned moments $\vec{m}_0$.

Response of electron spins and electron orbits to an applied magnetic field

Above we showed that an electron in circular orbit may have orbital or spin contributions, however in many situations neither of these contributions is large. The orbital contributions may be small in s-states or if there are electrons in each of the possible high angular momentum states, such as p-states. Moreover, in most situations, the response of the electron spin to an applied field depends on whether the atom or molecule has paired or unpaired electrons in each orbital. If the electrons are paired, the spin response is weak as the paired electrons must be broken up in order for the spins to align with an applied field. In contrast if there are unpaired electrons in an atom or molecule, the unpaired electrons can easily line up with an applied field. Paramagnetism and ferromagnetism occur when electron spins line up in a common direction. In the case of paramagnetism, an applied field is used to line up the spins. In ferromagnetic materials the spins can remain lined up even when an applied field is switched off.

Orbital contributions to the magnetic moment are almost always zero until the atom or molecule is placed in an applied field. The qualitative effect is that an applied magnetic field produces a diagmagnetic effect where the induced dipole moment is opposite to the original direction of the dipole in a circular orbit. This can be seen by considering the relationship between radius and velocity for the Bohr atom, ie

$$\frac{mv^2}{r} = \frac{ke^2}{r^2}; \quad \text{or} \quad vr = \frac{ke^2}{mv} \tag{7}$$

Now consider applying a small magnetic field that does not significantly alter this relationship, though of course a magnetic force $q\vec{v} \wedge \vec{B}$ does occur. The orbital moment is given by $-evr/2$ so increasing the product vr leads to an increase in the orbital moment. As seen from the equation above, smaller velocities (larger orbits) leads to larger orbital magnetic moment of course this is consistent with Bohr theory as vr is proportional to the magnitude of the angular momentum.

However the most strongest diagmagnetism occurs in superconductors where the effect can be perfect so as to screen out an applied field completely. In these materials superconducting screening currents are set up to completely cancel an applied magnetic field. In that sense they are qualitatively similar to the perfect dielectric behavior of metals where static charges perfectly screen out an applied electric field. As we shall see below there are some important differences that are not evident in these qualitative analogies.

The bottom line is that most atoms and molecules that have unpaired electrons are likely to be dominated by paramagnetic response, while those that have an even number of electrons, assuming that the electrons are paired, are typically diamagnetic. Superconductors are a special case where diamagnetism can be very strong.

Maxwell's equations for magnetic materials

We would like to extend Maxwell's equations to treat the magnetic fields inside and outside magnetic materials so we would like to replace the magnetization with current sources. It turns out that the relations between bound currents and the magnetization are,

$$\vec{K}_b = \vec{M} \wedge \vec{n}; \quad \vec{j}_b = \vec{\nabla} \wedge \vec{M} \tag{8}$$

If the magnetization inside a material is a constant, only surface currents are required to produce the magnetization, while if the magnetization varies in space, a bulk bound current density is also needed. The physical reasoning leading to these observations is constructed by taking local current loops and showing that if the local current loop density is a constant only a surface current remains. The formal calculation of these results is as follows.

The vector potential due to a magnetized material is found by adding up the vector potential contribution from each part of the material, so that,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r}') \wedge (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\tau' = \frac{\mu_0}{4\pi} \int \vec{M}(\vec{r}') \wedge \vec{\nabla}'(\frac{1}{|\vec{r} - \vec{r}'|}) d\tau' \tag{9}$$

Product rule 7 of Griffiths,

$$\vec{\nabla}(f\vec{A}) = f(\vec{\nabla} \wedge \vec{A}) - \vec{A} \wedge \vec{\nabla} f \tag{10}$$

leads to,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int (\frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \wedge \vec{M}(\vec{r}')) d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}' \wedge (\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|}) d\tau' \tag{11}$$

Using the vector identity (see Problem 1.60b of Griffiths)

$$\int (\vec{\nabla} \wedge \vec{A}) d\tau' = -\oint \vec{A} \wedge d\vec{a} \tag{12}$$

we finally have,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int (\frac{1}{|\vec{r} - \vec{r}'|} \vec{\nabla}' \wedge \vec{M}(\vec{r}')) d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{|\vec{r} - \vec{r}'|} (\vec{M}(\vec{r}') \wedge \hat{n}') da' \tag{13}$$

The first term is a superposition formula for bulk currents, provided $\vec{j}_b = \vec{\nabla} \wedge \vec{M}$, while the second integral corresponds to a superposition of surface currents provided we define $\vec{K}_b = \vec{M} \wedge \hat{n}$.

Incorporating bound currents into Ampere's law

If we have a problem where we know the magnetization, we can solve for the magnetic fields using Ampere's law where,

$$\vec{\nabla} \wedge \vec{B} = \mu_0(\vec{j}_f + \vec{j}_b) = \mu_0(\vec{j}_f + \vec{\nabla} \wedge \vec{M}) = \mu_0(\vec{\nabla} \wedge \vec{H} + \vec{\nabla} \wedge \vec{M}) \tag{14}$$

where we defined $\vec{H}$ to be the magnetic intensity and it obeys,

$$\vec{\nabla} \wedge \vec{H} = \vec{j}_f \quad \text{so that} \quad \oint \vec{H} \cdot d\vec{l} = i_f. \tag{15}$$

From this equation it is seen that Amperian contours can be used to relate i_f to the magnetic intensity (or Auxilliary field) $\vec{H}$. The magnetic intensity and magnetization have units Amp per meter (A/m) as can be seen from the definition of $\vec{H}$. Once the magnetic intensity has been found, the magnetic field is simply,

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \tag{16}$$

A uniformly magnetized cylinder with length >> radius

We take the magnetization to be uniform, $\vec{M} = M_0\hat{z}$ and to be oriented along the axis of the cylinder. There are no free currents, so there is no gain in using the magnetic intensity ($H = 0$ as $i_f = 0$) and $\vec{B} = \mu_0\vec{M}$. The magnetization is uniform, so the curl of the magnetization is zero, therefore the only bound currents are the surface currents $\vec{K}_b$. Since the magnetic field and magnetization are uniform, we can think of the problem as being similar to the the uniform field inside a solenoid, so we expect to find bound surface currents flowing around the cylinder. We find the suface currents through,

$$\vec{K}_b = M_0\hat{z} \wedge \hat{s} = M_0\hat{\phi} \tag{17}$$

This result applies in central regions of the magnetized cylinder. If we now consider a point $\vec{r}$ which is well removed from the cylinder, i.e. $r >> l$, then we can treat the cylinder as a dipole with dipole moment $\vec{m}_{cyl} = Volume \times \vec{M}$, and then find the field using the dipole formula,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_{cyl} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \tag{18}$$

As a check, lets use Ampere's law to find the magnetic field inside the cylinder. An amperian loop leads to $Bl = \mu_0 K_b l$, so that $\vec{B} = \mu_0 K_b$.

A Uniformly magnetized sphere of radius R

Take the magnetization to be, $\vec{M} = M_0\hat{z}$, so that the bound currents are now,

$$\vec{K}_b = \vec{M} \wedge \hat{r} = M_0 sin\theta\hat{\phi} \tag{19}$$

Assume that the magnetic field inside is uniform and along the $\hat{z}$ direction, and that outside it is like a dipole, so that,

$$\vec{B}^{int} = B_0\hat{z}; \quad \text{and} \quad \vec{B}^{ext} = \frac{\mu_0}{4\pi} \frac{3(\vec{m}_{sphere} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \tag{20}$$

The boundary conditions on the magnetic field at the sphere surface are, $B_t^{ext} - B_t^{int} = \mu_0 K_b$, $B_n^{ext} = B_n^{int}$. The latter equation gives,

$$B_0\hat{z} \cdot \hat{r} = B_0 cos\theta = \frac{\mu_0}{4\pi} \frac{2\vec{m}_{sphere} \cdot \hat{r})}{R^3} = \frac{\mu_0}{4\pi} \frac{2m_{sphere} cos\theta}{R^3} \tag{21}$$

From this expression, we find that

$$B_0 = \frac{\mu_0}{4\pi} \frac{2M_0 4\pi R^3}{3R^3} = \frac{2\mu_0 M_0}{3} \tag{22}$$

and hence,

$$\vec{B}_{inside} = \frac{2\mu_0 M_0}{3}\hat{z} \tag{23}$$

It is easy to show that the boundary condition on the tangential component is also satisfied, so we have the correct solution. Notice that this solution differs by a factor of two from the analagous solution for a uniformly polarization sphere.