PHY481 - Lecture 3: Vector calculus
Griffiths: Chapter 1 (Pages 38-54), Also Appendix A of Griffiths

Vector calculus in three co-ordinate systems

We shall be using three orthogonal co-ordinate systems, cartesian, cylindrical and spherical polar that are defined as follows

1. Cartesian co-ordinates

$$\vec{r} = (x, y, z) = x\hat{x} + y\hat{y} + z\hat{z} \tag{1}$$

2. Cylindrical co-ordinates

$$\vec{r} = (s, \phi, z) = s\hat{s} + z\hat{z} = scos(\phi)\hat{x} + ssin(\phi)\hat{y} + \hat{z} \tag{2}$$

where,

$$x = scos(\phi); \quad y = ssin(\phi); \quad x^2 + y^2 = s^2 \tag{3}$$

3. Spherical polar co-ordinates

$$\vec{r} = (r, \theta, \phi) = r\hat{r} = rsin(\theta)cos(\phi)\hat{x} + rsin(\theta)sin(\phi)\hat{y} + rcos(\theta)\hat{z} \tag{4}$$

where,

$$x = rsin(\theta)cos(\phi); \quad y = rsin(\theta)sin(\phi); \quad z = rcos(\theta); \quad x^2 + y^2 + z^2 = r^2 \tag{5}$$

Unit vector transformations

Relations between unit vectors in the three co-ordinate systems are very useful and can be derived using geometric reasoning. First consider cylindrical co-ordinates. It is easy to see that,

$$\hat{s} = cos(\phi)\hat{x} + sin(\phi)\hat{y}; \quad \hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y} \tag{6}$$

The first of these equations follows from Eq. (2) and the second is evident on taking projections of $\hat{\phi}$ onto the x and y directions. It is also easy to find the transformations from cylindrical to cartesian unit vectors;

$$\hat{x} = cos(\phi)\hat{r} - sin(\phi)\hat{\phi}; \quad \hat{y} = sin(\phi)\hat{r} + cos(\phi)\hat{\phi} \tag{7}$$

The transformations in the case of polar co-ordinates are,

$$\hat{r} = sin(\theta)cos(\phi)\hat{x} + sin(\theta)sin(\phi)\hat{y} + cos(\theta)\hat{z}; \tag{8}$$

which follows from Eq. (4),

$$\hat{\theta} = cos(\theta)cos(\phi)\hat{x} + cos(\theta)sin(\phi)\hat{y} - sin(\theta)\hat{z}; \tag{9}$$

and

$$\hat{\phi} = -sin(\phi)\hat{x} + cos(\phi)\hat{y} \tag{10}$$

which is the same as the second of Eq. (6). The equations relating unit vectors in spherical polars to those in cartesian co-ordinates are,

$$\hat{x} = sin(\theta)cos(\phi)\hat{r} + cos(\theta)cos(\phi)\hat{\theta} - sin(\phi)\hat{\phi}; \tag{11}$$

$$\hat{y} = sin(\theta)sin(\phi)\hat{r} + cos(\theta)sin(\phi)\hat{\theta} + cos(\phi)\hat{\phi}; \tag{12}$$

and

$$\hat{z} = cos(\theta)\hat{r} - sin(\theta)\hat{\theta} \tag{13}$$

Curvilinear co-ordinates: Scale factors h_1, h_2, h_3

In general a set of curvilinear co-ordinates can be orthogonal or non-orthogonal. We focus on the orthogonal case, which includes cartesian, cylindrical and spherical co-ordinates. We denote the unit vectors as $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and a position vector $\vec{l}$ is written as,

$$\vec{l} = (u_1, u_2, u_2) = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3. \tag{14}$$

For the Cartesion, Cylindrical, and Spherical Polar cases, we may write $(u_1, u_2, u_2) = (x, y, z), (s, 0, z), (r, 0, 0)$ respectively.

Now imagine displacing the co-ordinates by a small amount du_1, du_2, du_3, this leads to a change in the vector $\vec{l}$ by an amount $d\vec{l}$. In general we can write

$$d\vec{l} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \tag{15}$$

where h_1, h_2, h_3 are the scale factors. They are central to deriving the relations that we need. For the three cases of interest in this course, we have,

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}; \quad d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z} \quad d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + rsin(\theta)d\phi\hat{\phi} \tag{16}$$

leading to: For *cartesian co-ordinates*, $h_x = 1, h_y = 1, h_z = 1$; for *cylindrical co-orindates*, $h_s = 1, h_\phi = s, h_z = 1$; and for *spherical polar co-ordinates*, $h_r = 1, h_\theta = r, h_\phi = rsin\theta$.