PHY481 - Lecture 4: Vector calculus in curvilinear co-ordinates

Griffiths: Chapter 1 (Pages 38-54), Also Appendix A of Griffiths

Scale factors h_1, h_2, h_3

In general a set of curvilinear co-ordinates can be orthogonal or non-orthogonal. We focus on the orthogonal case, which includes cartesian, cylindrical and spherical co-ordinates. We denote the unit vectors as $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and a position vector $\vec{l}$ is written as,

$$\vec{l} = (u_1, u_2, u_2) = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3. \tag{1}$$

For the Cartesion, Cylindrical, and Spherical Polar cases, we may write $(u_1, u_2, u_2) = (x, y, z), (s, 0, z), (r, 0, 0)$ respectively.

Now imagine displacing the co-ordinates by a small amount du_1, du_2, du_3, this leads to a change in the vector $\vec{l}$ by an amount $d\vec{l}$. In general we can write

$$d\vec{l} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \tag{2}$$

where h_1, h_2, h_3 are the scale factors. They are central to deriving the relations that we need. For the three cases of interest in this course, we have,

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}; \quad d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z} \quad d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + rsin(\theta)d\phi\hat{\phi} \tag{3}$$

leading to: For *cartesian co-ordinates*, $h_x = 1, h_y = 1, h_z = 1$; for *cylindrical co-ordinates*, $h_s = 1, h_\phi = s, h_z = 1$; and for *spherical polar co-ordinates*, $h_r = 1, h_\theta = r, h_\phi = rsin\theta$.

Volume element, grad, div, curl in the three co-ordinate systems

1. A *volume element* dV (or $d\tau$ in Griffiths) at position $\vec{r}$ is given by,

$$d\tau = h_1 h_2 h_3 du_1 du_2 hu_3. \tag{4}$$

This is simply the volume of a cube centered at position $\vec{r}$, with sides h_1du_1, h_2du_2, h_3du_3. In cartesian co-ordinates we then have $d\tau = dxdydz$, while in cylindrical co-ordinates $d\tau = sdsd\phi dz$ and in spherical polar co-ordinates, $d\tau = r^2 sin(\theta) dr d\theta d\phi$. Most of Assignment 1 is doing integrals, particularly to confirm Stoke's theorem and the divergence theorem, ie.

$$\oint \vec{v} \cdot d\vec{l} = \int (\vec{\nabla} \wedge \vec{v}) \cdot d\vec{a}; \quad \oint \vec{v} \cdot d\vec{a} = \int (\vec{\nabla} \cdot \vec{v}) d\tau \tag{5}$$

For many of these problems it is best to use cylindrical or spherical polar co-ordinates, so we need expressions for grad, div and curl in these co-ordinate systems. These expressions can be deduced using a general orthogonal curvilinear description.

1. The gradient is found by using two forms for the expansion of a scalar function

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3 \tag{6}$$

and

$$df = d\vec{l} \cdot \vec{\nabla} f = dl_1 (\vec{\nabla} f)_1 + dl_2 (\vec{\nabla} f)_2 + dl_3 (\vec{\nabla} f)_3 = h_1 du_1 (\vec{\nabla} f)_1 + h_2 du_2 (\vec{\nabla} f)_2 + h_3 du_3 (\vec{\nabla} f)_3 \tag{7}$$

Where we used the definition of $d\vec{l}$ (Eq. (15) to write,

$$dl_1 = h_1 du_1; \quad dl_2 = h_2 du_2 \quad dl_3 = h_3 du_3. \tag{8}$$

Combining Eqs. (13) and (15), yields,

$$\vec{\nabla} = (\frac{1}{h_1}\frac{\partial}{\partial u_1}, \frac{1}{h_2}\frac{\partial}{\partial u_2}, \frac{1}{h_3}\frac{\partial}{\partial u_3}) \tag{9}$$

For example if we have an electrostatic potential V calculated in spherical polar co-ordinates, we can find the electric field through $\vec{E} = -\vec{\nabla} V$ using the expressions for the scale factors in spherical polar co-ordinates.

2. The divergence is found by considering a small cube of dimension ϵ. Consider the surface integral of the flux through this cube in cartesian co-ordinates, that is,

$$\oint_{dS} \vec{v} \cdot \hat{n} da = \sum_{i=1}^{3} [v_i(\vec{x} + \epsilon \hat{e}_i/2) - v_i(\vec{x} - \epsilon \hat{e}_i/2)]\epsilon^2 \tag{10}$$

where here we use $(\hat{e}_1, \hat{e}_2, \hat{e}_3) = (\hat{x}, \hat{y}, \hat{z})$, and $x_1 = x, x_2 = y, x_3 = z$. Eq. (17) may be written as,

$$\oint_{dS} \vec{v} \cdot \hat{n} da = \sum_{i=1}^{3} \frac{\partial v_i(\vec{x})}{\partial x_i} \epsilon^3 = (\vec{\nabla} \cdot \vec{v})\epsilon^3, \tag{11}$$

so that a co-ordinate independent expression for the divergence of a vector function is,

$$\vec{\nabla} \cdot \vec{v} = lim_{\tau \to 0} \frac{1}{\tau} \oint_{dS} \vec{v} \cdot d\vec{a}. \tag{12}$$

The divergence is clearly proportional to the flux of the function, $\vec{v}$ though the surface of the volume τ. The divergence theorem is the generalization of this expression to finite volumes. The extension to any closed surface is understood by dividing the large volume into boxes and noting that the only flux that remains is the flux through the outer surfaces. The flux across the interior surfaces of the boxes cancel and the normal to interior planes takes on the two possible directions, canceling in the total integral on the RHS. The generalization to arbitrary curvilinear co-ordinates is,

$$\vec{\nabla} \cdot \vec{v} d\tau = [v_1 h_2 h_3|_{u_1+du_1} - v_1 h_2 h_3|_{u_1}] du_2 du_3 +$$

$$[v_2 h_1 h_3|_{u_2+du_2} - v_2 h_1 h_3|_{u_2}] du_1 du_3 + [v_3 h_1 h_2|_{u_3+du_3} - v_3 h_1 h_2|_{u_3}] du_1 du_2 \tag{13}$$

From this expression and using $dV = h_1 h_2 h_3 du_1 du_2 du_3$ we find,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} [\frac{\partial}{\partial u_1}(v_1 h_2 h_3) + \frac{\partial}{\partial u_2}(v_2 h_1 h_3) + \frac{\partial}{\partial u_3}(v_3 h_1 h_2)] \tag{14}$$

The expressions for the divergence of a vector funtion in cylindrical and spherical polar co-ordinates follows from this expression.

3. The Laplacian of a scalar function f is $\vec{\nabla} \cdot (\vec{\nabla} f)$, which reduces to,

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} [\frac{\partial}{\partial u_1}(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial u_3})] \tag{15}$$

4. The Curl is developed in a similar way. Consider a square loop placed in the x-y plane, with edge length ϵ. Consider a path integral around the loop, the circulation,

$$\oint_{loop} \vec{F} \cdot d\vec{l} = \epsilon F_i(\vec{x} - \epsilon \hat{e}_j/2) + \epsilon F_j(\vec{x} + \epsilon \hat{e}_i/2) - \epsilon F_i(\vec{x} + \epsilon \hat{e}_j/2) - \epsilon F_j(\vec{x} - \epsilon \hat{e}_i/2) \tag{16}$$

This reduces to,

$$\oint_{loop\ in\ xy\ plane} \vec{F} \cdot d\vec{l} = (\vec{\nabla} \wedge \vec{F})_z \epsilon^2 \tag{17}$$

The general co-ordinate independent form of $\vec{\nabla} \wedge \vec{F}$ is then

$$\hat{n} \cdot (\vec{\nabla} \wedge \vec{F}) = lim_{a \to 0} \frac{1}{a} \oint_C \vec{F} \cdot d\vec{l} \tag{18}$$

The proof of Stokes theorem is the finite volume generalization of this expression and follows from breaking up the finite domain into small squares and noting that all interior loops segments have zero net contribution to the contour integral, due to the opposite sense of the circulation in wires shared by two squares. The generalization of this expression to general curvilinear co-ordinates yields,

$$\vec{\nabla} \wedge \vec{F} = \begin{vmatrix} \hat{e}_1/h_2h_3 & \hat{e}_2/h_3h_1 & \hat{e}_3/h_1h_2 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \quad (19)$$

Helmholtz theorem. Any vector field $\vec{F}$ with the properties

$$lim_{r\to\infty}\vec{\nabla}\cdot\vec{F} \to 0, \quad lim_{r\to\infty}\vec{\nabla}\wedge\vec{F} \to 0,$$

may be broken up into a divergence free part and an irrotational (curl free) part, so that,

$$\vec{F} = \vec{G}(curlfree - irrotational) + \vec{H}(divergencefree - solenoidal) = -\vec{\nabla}V + \vec{\nabla}\wedge\vec{A} \quad (20)$$

Two important consequences of this theorem are as follows. An irrotational, curl free, function obeys $\vec{\nabla}\wedge\vec{G} = 0$. Since we have the identity $\vec{\nabla}\wedge(\vec{\nabla}V) = 0$ where V is a scalar function, it is natural to define $\vec{G} = -\vec{\nabla}V$. The minus sign is written explicitly as we will be using this in electrostatics where $\vec{G} = \vec{E}$. In a similar way the vector identity $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{A}) = 0$ implies that the solenoidal function $\vec{H} = \vec{\nabla}\wedge\vec{A}$. Notice that there is no minus sign in this expression, as is conventional in magnetostatics where $\vec{H} = \vec{B}$ and $\vec{A}$ is the vector potential.

Delta function

The delta function $\delta(x)$ is a very useful function in physics and mathematics. It is a function that is very sharply peaked when its argument is zero (i.e. $x = 0$) and it is zero everywhere else. The delta function can be defined as the limit of many different functions. One very useful definition is by using Gaussians, so that,

$$\delta(x) = lim_{\alpha\to\infty}(\frac{\alpha}{\pi})^{1/2}e^{-\alpha x^2} \quad (21)$$

This function is zero for any $x \neq 0$ and has the property that $\int_{-\infty}^{\infty}\delta(x)dx = 1$, as can be confirmed for the explicit function above. Other useful ways of defining the delta function are through a Lorentian or through a Fourier decomposition both of which are used extensively in theoretical many body physics. Delta functions are often used to impose constraints. If we want to specify a position in three dimensions, we write $\delta(\vec{r})$, which really means $\delta(x)\delta(y)\delta(z)$.

In electrostatics it is often to use continuous functions instead of discrete sums and the delta function allows us to change from sums to integrals. For example a charge, Q, at position $\vec{r}$ can be written in terms of the continous charge density $\rho(\vec{r}') = Q\delta(\vec{r}-\vec{r}'$. To confirm that this is correctly normalized, integrate both sides over $\vec{r}'$. A second example is where we want to specify a distributed charge, for example a charge Q uniformly distributed on the perimeter of a circle of radius s. In that case we used cylindrical co-ordinates to write $\rho(\vec{r}') = A\delta(s'-s)\delta(z)$. Integrating both sides, the LHS gives Q, while the RHS gives $2\pi s$. We then have $A = Q/(2\pi s)$