PHY481 - Lecture 5: Electrostatics
Griffiths: Chapter 2

The basics: Coulomb's law, the electric field and superposition

Electrostatics is based on very simple principles, namely that (i) there are two types of charge, (ii) that charges interact through Coulomb's law i.e.,

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^3}\vec{r} \tag{1}$$

and (iii) that superposition holds. We also have to distinguish between conductors (metals) and insulators (dielectrics).

To become proficient in electrostatics we need to introduce some more advanced ideas and methods. The first advanced idea is to define the electric field due to a charge Q through,

$$\vec{E} = \frac{\vec{F}}{q}, \tag{2}$$

so that the electric field is the force per unit charge. Since the unit of force is the Newton (N), the unit of electric field is N/C, where C is the unit of electric charge, the Coulomb. $k = 1/4\pi\epsilon_0 = 9\times 10^9 Nm^2/C^2 = 9\times 10^9 kgm^3/C^2s^2$.

We also want a more general form for the electric field at position $\vec{r}$, due to a point charge at position $\vec{r}'$, so we write the electric field due to a point charge in the important more general form,

$$\vec{E}(\vec{r}, \vec{r}') = k\frac{Q}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}') \tag{3}$$

This form of the electric field is important as it explicitly contains the location of the "source", ie. $\vec{r}'$. Since the superposition principle holds for the force, it also applies to the electric field, so the electric field at position $\vec{r}$ due to n charges at positions $\vec{r}_1...\vec{r}_n$ is given by a sum over terms like that in Eq. (3), i.e.,

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} k\frac{Q_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i) = k\int \frac{d\vec{r}'\rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3}(\vec{r} - \vec{r}') \tag{4}$$

where $\rho(\vec{r}') = \sum_i Q_i\delta(\vec{r}' - \vec{r}_i)$. These are the two basic forms of the superposition law for the electric field and are one of the two main approaches discussed in Chapter 2 of Griffiths. When treating continuous charge distributions, we may be given a charge per unit length, λ, a charge per unit area σ or a charge per unit volume ρ.

Typical superposition problems

Example 1 - discrete charge distribution. Find the electric field due to six charges, q, on the corners of a hexagon that has sides of length a. Take the hexagon to lie in the x-y plane, centered at the origin and with its long axis along the x-direction. Find the electric field on the x-axis.

Solution. Lets calculate on the positive x-axis, with $x > a$. The locations of the charges are: $\vec{s}_1 = (a, 0, 0)$, $\vec{s}_2 = (a/2, 3^{1/2}a/2, 0)$, $\vec{s}_3 = (a/2, -3^{1/2}a/2, 0)$, $\vec{s}_4 = (-a/2, 3^{1/2}a/2, 0)$, $\vec{s}_5 = (-a/2, -3^{1/2}a/2, 0)$, $\vec{s}_6 = (-a, 0, 0)$. By symmetry the electric field on the x-axis is $\vec{E} = (E_x, 0, 0)$. By superposition, we then have,

$$E_x(x > a) = kq(\frac{1}{r_1^2} + \frac{1}{r_6^2} + \frac{2cos(\theta_2)}{r_2^2} + \frac{2cos(\theta_4)}{r_4^2}) \tag{5}$$

From the geometry of the problem we have,

$$r_1^2 = (x - a)^2; \quad r_6^2 = (x + a)^2; \quad r_2^2 = (x - a/2)^2 + 3a^2/4;$$

and

$$r_4^2 = (x + a/2)^2 + 3a^2/4; \quad cos(\theta_2) = (x - a/2)/r_2; \quad cos(\theta_4) = (x + a/2)/r_4. \tag{6}$$

The electric field $E_x(x < -a) = -E_x(x > a)$. We have to be a little more careful when $-a < x < a$ as the signs of the four contributions are no longer all positive. This is left as an exersize.

Example 2. - continuous charge distribution. Find the electric field on the z-axis for a uniformly charged ring, with linear charge density λ and radius R, centered at the origin and lying in the x-y plane.

Solution. By symmetry, $\vec{E} = (0, 0, E_z)$, and by superposition we have,

$$E_z = \int \frac{kdQ(\phi)}{r^2} cos(\theta) = \int_0^{2\pi} k\frac{\lambda s d\phi}{r^2} cos(\theta) = \frac{kQz}{(R^2 + z^2)^{3/2}} \tag{7}$$

where we used,

$$s = R; \quad Q = 2\pi\lambda; \quad r^2 = R^2 + z^2; \quad cos(\theta) = z/r. \tag{8}$$

Electric field lines - Faraday's ideas

An extremely useful concept in developing new ideas and results in EM is the concept of electric field lines. An electric field line is a series of vectors where at each point the vector points in the direction of the force on a unit charge at that point and it has a length equal to the magnitude of the force. ie. we plot the vector function $\vec{E}$. The properties of electric field lines constructed in this way are as follows.

1. At each point along an electric field line, the force on a positive test charge is in a direction tangent to the field line at that point. This implies that electric field lines come out of positive charges and go into negative charges.

2. The density of lines at any point in space is proportional to the magnitude of the electric field at that point.

3. Electric field lines begin and/or end at charges, or they continue off to infinity. i.e. they do not begin or end in free space.

4. Electric field lines do not cross.

Very important special case: conductors

1. If there is no current flowing, then the electric field is zero, $\vec{E} = 0$, inside a conductor.

2. If there is no current flowing, then at the surface of a conductor, the electric field is normal to the surface of the conductor.

Gauss's Law

The integral form of Gauss's law in free space is,

$$\text{Electric flux} = \phi_E = \oint_S \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0} = \frac{q}{\epsilon_0} \tag{9}$$

where q_{encl} is the total charge inside the closed surface S, and usually we will replace it by q, with the fact that it is the enclosed charge taken implicitly. This law follows from Coulomb's law and superposition in combination with the properties of electric field lines. The proof of Gauss's law in general follows from the following statements.

Property 1. (i) For a charge q with a spherical shell at radius r it is easy to prove that Gauss's law is correct. (ii) For a charge that is outside of a spherical shell it is also easy to prove that the total flux through the shell is zero. This is proven by noting all flux lines originate or terminate at a charge, or go to infinity. Therefore a flux line originating from a charge outside the spherical shell and which enters the spherical shell must also leave the spherical shell. The net flux through the surface of the shell due to that flux line is then zero.

Property 2. Flux lines are like a conserved fluid flow so that any surface drawn around a charge must have the same total flux through it. This gives us a great deal of freedom in drawing the surfaces through which flux flows. For a single charge clearly a spherical surface is most convenient.

Property 3. If we have a distribution of charge inside a Gaussian surface, we can break the charge up into small pieces and treat each piece with a spherical surface, and the total flux through a surface surrounding all of the charges is the same as the sum of the flux due to each little piece of charge through a spherical surface surrounding that charge. This property is due to the superposition property that is correct in electrostatics.

The q_{encl} on right hand side of Eq. (9) is the total charge in the volume τ enclosed by the closed surface S, which may be written in either discrete or continuous forms,

$$q_{encl} = \sum_{i\varepsilon\tau} q_i = \int_\tau \rho(\vec{r})d\vec{r} \tag{10}$$

where $\rho(\vec{r})$ is the charge density.

The differential form of Gauss's law follows from using the last expression on the RHS of this equation and applying the divergence theorem to the LHS of Eq. (9) to find,

$$\oint_S \vec{E} \cdot d\vec{a} = \int_\tau (\vec{\nabla} \cdot \vec{E})d\tau = \frac{1}{\epsilon_0} \int_\tau \rho d\tau \quad \text{which implies} \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \tag{11}$$

Note that the superposition integral gives the solution to electrostatics and must satisfy the differential forms, ie.

$$If \quad \vec{E}(\vec{r}) = \int_\tau [\frac{k\rho(\vec{r'})d\vec{r'}}{|\vec{r} - \vec{r'}|^3}(\vec{r} - \vec{r'})] \quad \text{then} \quad \vec{\nabla} \wedge \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \tag{12}$$