

PHY481 - Lecture 6: Gauss's law

Griffiths: Chapter 2

Electric field lines - Faraday's ideas

An extremely useful concept in developing new ideas and results in EM is the concept of electric field lines. An electric field line is a series of vectors where at each point the vector points in the direction of the force on a unit charge at that point and it has a length equal to the magnitude of the force. ie. we plot the vector function \vec{E} . The properties of electric field lines constructed in this way are as follows.

1. At each point along an electric field line, the force on a positive test charge is in a direction tangent to the field line at that point. This implies that electric field lines come out of positive charges and go into negative charges.
2. The density of lines at any point in space is proportional to the magnitude of the electric field at that point.
3. Electric field lines begin and/or end at charges, or they continue off to infinity. i.e. they do not begin or end in free space.
4. Electric field lines do not cross.

Very important special case: conductors

1. If there is no current flowing, then the electric field is zero, $\vec{E} = 0$, inside a conductor.
2. If there is no current flowing, then at the surface of a conductor, the electric field is normal to the surface of the conductor.

Gauss's Law

The integral form of Gauss's law in free space is,

$$\text{Electric flux} = \phi_E = \oint_S \vec{E} \cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0} = \frac{q}{\epsilon_0} \quad (1)$$

where q_{encl} is the total charge inside the closed surface S , and usually we will replace it by q , with the fact that it is the enclosed charge taken implicitly. This law follows from Coulomb's law and superposition in combination with the properties of electric field lines. The proof of Gauss's law in general follows from the following statements.

Property 1. (i) For a charge q with a spherical shell at radius r it is easy to prove that Gauss's law is correct. (ii) For a charge that is outside of a spherical shell it is also easy to prove that the total flux through the shell is zero. This is proven by noting all flux lines originate or terminate at a charge, or go to infinity. Therefore a flux line originating from a charge outside the spherical shell and which enters the spherical shell must also leave the spherical shell. The net flux through the surface of the shell due to that flux line is then zero.

Property 2. Flux lines are like a conserved fluid flow so that any surface drawn around a charge must have the same total flux through it. This gives us a great deal of freedom in drawing the surfaces through which flux flows. For a single charge clearly a spherical surface is most convenient.

Property 3. If we have a distribution of charge inside a Gaussian surface, we can break the charge up into small pieces and treat each piece with a spherical surface, and the total flux through a surface surrounding all of the charges is the same as the sum of the flux due to each little piece of charge through a spherical surface surrounding that charge. This property is due to the superposition property that is correct in electrostatics.

The q_{encl} on right hand side of Eq. (1) is the total charge in the volume τ enclosed by the closed surface S , which may be written in either discrete or continuous forms,

$$q_{encl} = \sum_{i \in \tau} q_i = \int_{\tau} \rho(\vec{r}') d\vec{r}' \quad (2)$$

where $\rho(\vec{r}')$ is the charge density.

The differential form of Gauss's law follows from using the last expression on the RHS of this equation and applying the divergence theorem to the LHS of Eq. (1) to find,

$$\oint_S \vec{E} \cdot d\vec{a} = \int_{\tau} (\vec{\nabla} \cdot \vec{E}) d\tau = \frac{1}{\epsilon_0} \int_{\tau} \rho d\tau \quad \text{which implies} \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad (3)$$

Note that the superposition integral gives the solution to electrostatics and must satisfy the differential forms, ie.

$$\text{If } \vec{E}(\vec{r}) = \int_{\tau} \left[\frac{k\rho(\vec{r}')d\vec{r}'}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \right] \text{ then } \vec{\nabla} \wedge \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad (4)$$

Solving electrostatics problems using Gauss's law

Gauss's law is only useful for exact solutions when the electric field has high symmetry, typically either spherical, cylindrical or planar symmetry. The symmetry of the electric field follows from the symmetry of the charge distribution that we are given.

1. *Spherical symmetry:* In this case we consider shell's of charge or even spheres of charge where the charge density only depends on r . Consider first a uniform shell of charge of radius R , with charge per unit area σ (with units C/m^2 of course). Find the electric field for $r < R$ and $r > R$. First we note that "by symmetry" the electric field only depends on r , so we can write $\vec{E}(\vec{r}) = E(r)\hat{r}$. We therefore choose a spherical surface and on that surface $E(r)$ is a constant. The flux integral can then be carried out, so that,

$$\phi_E = \oint E(r)\hat{r} \cdot \hat{r} da = \int_0^\pi \int_0^{2\pi} E(r)r^2 \sin(\theta) d\theta d\phi = 4\pi r^2 E(r) \quad (5)$$

The charge enclosed by the Gaussian surface depends on the radius r : For $r < R$, $q_{encl} = 0$; while for $r > R$, $q_{encl} = 4\pi R^2 \sigma = Q$. we then find that $E(r) = kQ/r^2$ for $r > R$ and $E(r) = 0$ for $r < R$. This is the famous shell theorems, which in workds state that the electric field inside a uniform shell of charge is zero and outside is like a point charge at the origin. Now consider two uniform shells of charge of radii R_1 and R_2 , having charge densities σ_1 and σ_2 . Find the electric field in the three regimes $r < R_1$, $R_1 < r < R_2$, and $r > R_2$. Since we have spherical symmetry we of course have Eq. (5) so we only need to find the enclosed charge. In the three cases we have $q_{encl} = 0, Q_1, Q_1 + Q_2$. Alternatively we could solve the problem by superposition i.e. add the electric fields due to the two shells of charge. Capacitors are the special case where $Q_1 = -Q_2$. What does the electric field look like in that case? We will return to this a little later. It is relatively straightforward to generalize to an arbitrary charge density $\rho(r)$.

2. *Cylindrical symmetry:* Here the most basic problem is a uniform cylindrical shell of charge or radius s_1 . We choose a cylindrical Gaussian surface so the electric field is $\vec{E} = E(s)\hat{s}$. The flux through the Gaussian surface is,

$$\phi_E = \oint E(s)\hat{s} \cdot \hat{s} da = \int_0^L \int_0^{2\pi} E(r) s d\phi dz = 2\pi s L E(s) \quad (6)$$

The enclosed charge is 0 for $s < s_1$, and $2\pi s_1 L \sigma$ for $s > s_1$, so the electric field is $E(s) = 0$ for $s < s_1$ and $E(s) = \lambda/(2\pi\epsilon_0 s)$ for $s > s_1$. Here it is nice to define the charge per unit length $\lambda = 2\pi s_1 \sigma$. Using a similar procedure or superposition it is straightforward to write down the electric field due to two concentric cylinders of charge, with radii s_1 and s_2 and charge per unit length λ_1 and λ_2 . Capacitors correspond to the case $\lambda_1 = -\lambda_2$. It is relatively straightforward to generalize to an arbitrary charge density $\rho(s)$.

3. *Planar symmetry:* Here we consider an infinite, uniform, thin sheet of charge lying in the x-y plane, with charge density σ per unit area. By symmetry the electric field is $\vec{E}(\vec{r}) = E(z)\hat{z}$. Now we can choose a cylindrical or rectangular Gaussian surface as the only flux is through the top and bottom surfaces. If we define a to be the area of the top and bottom surfaces, we find the flux to be,

$$\phi_E = \oint E(s)\hat{s} \cdot \hat{s} da = 2aE(z) \quad (7)$$

The electric field is then $\vec{E}(\vec{r}) = \sigma\hat{z}/2\epsilon_0$. The case of many parallel sheets of charge can be solved using superposition or by applying Gauss's theorem in a similar way. Capacitors correspond to the case of two sheets with $\sigma_1 = -\sigma_2$. What does the electric field look like in that case? What does the electric field look like if we consider a finite sheet of charge?

4. *The remarkable case of the electric field near a conducting surface*

There is a general and remarkable result for conducting surfaces that follows straightforwardly from the fact that $E_{||} = 0$ at conducting surfaces and that $\vec{E} = 0$ inside conductors. In that case (i) Charges inside conductors are "screened" and the net charge ends up at the surfaces of the conductor density. (ii) If there is a charge density $\sigma(\vec{r})$ at position \vec{r} at the surface of a conductor, then the electric field near \vec{r} and outside the conductor is $(\sigma(\vec{r})/\epsilon_0)\hat{n}$.