PHY481 - Lecture 8: Energy in a charge distribution, capacitance
Griffiths: Chapter 2

The potential energy of a charge distribution

The potential energy required to place a small charge q at position $\vec{r}$ is $U = qV(\vec{r})$. We can generalize this to a continuum form, however we must keep in mind that it is only correct if V does not change as charge is added, i.e.

$$U = \int \rho(\vec{r})V(\vec{r})d\vec{r} \qquad \text{charge added when the potential is fixed} \tag{1}$$

In cases where the potential does change as charge is added, we have to carry out a second integral or sum, as in the cases treated below.

Potential energy stored in a charge distribution

A different question is: What is the potential energy of a distribution of charges, that is, what is the potential energy stored in a distribution of charges. In this case we muts take into account the way in which the electrostatic potential changes as charge is added to the system. The potential energy stored in a distribution of charges is equal to the work done in setting up the distribution of charges, provided there is no dissipation and no kinetic energy is generated. To set up a distribution of charges Q_i at positions $\vec{r_i}$, we need to bring each of the charges in from infinity and place it at its allocated position. The work required to place the first charge is zero (no other charges are there yet). The work required to place the second charge is Q_2V_{21}, where $V_{21} = kQ_1/r_{21}$ is the electric potential at position $\vec{r_2}$ due to charge Q_1. Note that $r_{21} = r_{12} = |\vec{r}_2 - \vec{r}_1|$. The work required to place charge 3 at its position is equal to $Q_3V_{31} + Q_3V_{32}$, and so on, once all of the n charges are in position, we have,

$$U_n = \frac{1}{2}\sum_{i\neq j}^{n,n} \frac{kQ_iQ_j}{r_{ij}} = \sum_{i<j}^{n,n} \frac{kQ_iQ_j}{r_{ij}} \tag{2}$$

In these expressions each pair interaction is counted once and the total potential energy is the sum of the potential energies of all pairs. Note: in writing this energy we have ignored the self-energy of each charge. The self-energy is $n * E_{self}$ and is the same regardless of where the charges are places. U_n is the interaction energy between the charges. Taking the continuum limit the electric potential due to a charge distribution is,

$$U = \frac{1}{2}\int \frac{\rho(\vec{r})\rho(\vec{r}')d\vec{r}d\vec{r}'}{4\pi\epsilon_0|\vec{r}-\vec{r}'|} = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})d\vec{r} \quad \text{to set up charge distribution} \tag{3}$$

Note that this is 1/2 the value which would be true if the potential were fixed and when the charge $\int \rho(\vec{r})$ as added. This factor of two is thus quite fundamental - it is also a source of considerable confusion for some students.

The potential energy stored is stored in the electric field!

Here we show that the potential energy is stored in the electric field itself by writing Eq. (3) in a different form. First use Poisson's equation to write,

$$\rho V = -\epsilon_0(\nabla^2 V)V. \tag{4}$$

Then use the vector identity,

$$\vec{\nabla}\cdot(V\vec{E}) = \vec{\nabla}V\cdot\vec{E} + V\vec{\nabla}\cdot\vec{E}, \quad \text{or} \quad V\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(V\vec{E}) - \vec{\nabla}V\cdot\vec{E} \tag{5}$$

Using $E = -\vec{\nabla}V$, this leads to

$$\rho V = -\epsilon_0(\nabla^2 V)V = -\epsilon_0\vec{\nabla}\cdot(V\vec{\nabla}V) + \epsilon_0(\vec{\nabla}V)^2 \tag{6}$$

Using the divergence (Gauss's) theorem the volume integral of the term $\int \vec{\nabla}\cdot(V\vec{\nabla}V)d\tau$ becomes $\oint V\vec{\nabla}V\cdot d\vec{A}$ which goes to zero at $r\to\infty$. The only surviving term is the volume integral of the last term on the RHS, so that the energy density (energy per unit volume) in the electric field may then be written as,

$$u(\vec{r}) = \frac{1}{2}\epsilon_0\vec{E}^2 \tag{7}$$

where we used the fact that $\vec{E} = -\vec{\nabla}V$. This is the energy required to set up the charge distribution.

Note that if we integrate the field due to an isolated charge we get infinity!. However we are interested in changes in potential energy due to changing the charge configuration. The infinite self-energy of each charge regardless of how the charges are arranged, so it plays no role in the physics of the problem. We therefore omit it from all calculations.

The energy stored in the electric field of the earth is,

$$U_{earth\ field} \approx Volume * \frac{\epsilon_0}{2}(110V/m)^2 = (4\pi(6400km)^2 2km)\frac{\epsilon_0}{2}(110V/m)^2 = 4 * 10^{10} J \tag{8}$$

Is this a lot of energy? One gallon of gasolene has energy content is $1.26 \times 10^8 J$. The unit of energy used in our houses is kw-hr. $1kw - hr = 1000W \times 3600s = 3.6 \times 10^6 J$. We pay only about $11c/kw - hr$ so energy is relatively cheap.

In the above we use the fact that the earth's electric field reduces to about one half of its sea level value at an altitude of $2km$. There are considerable variations from place to place on the earth's surface and this is a rough average value, but good enough for an estimate correct to an order of magnitude or so.

Definition of capacitance

Capacitance measures the ability of a system to store charge. It is assumed that if the applied voltage is zero, no (net) charge is stored, and that when a voltage is applied charge starts to be stored. The basic geometry is set up by attaching one piece of metal to the positive lead of a battery, while another piece of metal is attached to the other electrode of a battery. The fundamental relation is then,

$$Q = CV \tag{9}$$

where V is the voltage of the battery and Q is the charge stored on the pieces of metal, $+Q$ on the piece of metal attached to the positive electrode of the battery and $-Q$ on the piece of metal attached to the negative electrode. Due to the fact that the potential difference is linear in the charge on the plates, we get the relation above. This linearity follows from any of the forms we have used to calculate the potential, e.g. the superposition formula where $\rho(\vec{r}')$ appears in a linear way. In general C is hard to calculate, however for highely symmetry cases where $\vec{E}$ or V can be calculated, we can find explicit formulae as follows:

Parallel Plates For two parallel plates that have charge Q and $-Q$, we have,

$$Q = \sigma A; \quad E = \frac{\sigma}{\epsilon_0}; \quad \text{and} \quad V = Ed = \frac{Qd}{A\epsilon} \tag{10}$$

Which can be written as,

$$Q = \frac{\epsilon_0 A}{d} V = CV \tag{11}$$

so that the capacitance of a parallel plate capacitor is $C_{plate} = \epsilon_0 A/d$ as you know.

Example 1. A coaxial cable

Consider two coaxial cylindrical metal shells of radii a and b, where $b > a$ and $d = b - a$. A voltage V is applied to the outer cylinder and the inner cylinder is grounded so the total charge on the two shells is Q on the outer and $-Q$ on the inner shell. The electric field for $s > b$ and for $s < a$ is zero by Gauss's theorem. The electric field for $a < s < b$ is found from Gauss's law,

$$E 2\pi s L = -Q/\epsilon_0 \quad \text{so that} \quad E(r) = -\frac{Q}{2\pi s L\epsilon_0} \tag{12}$$

Now we find the potential difference by integration,

$$V_b = -\int_a^b \vec{E} d\vec{l} = \int_a^b \frac{Q}{2\pi s L\epsilon_0} ds = \frac{Q}{2\pi L\epsilon_0} log(b/a) \tag{13}$$

The voltage difference is $V = V_b$ as $V_a = 0$ because the inner cylinder is grounded. Note also that we are not using the usual rule that the potential at infinity is zero in this calculation. Anyway all that matters is the potential difference and from Eq. (11)

$$Q = \frac{2\pi L\epsilon_0}{log(b/a)} V, \tag{14}$$

so that $C_{coax} = 2\pi L\epsilon_0/log(b/a)$

Example 2. Concentric spherical shells.

Consider two concentric metal shells of radii a and b, where $b > a$ and $d = b - a$. A voltage V is applied to the outer cylinder and the inner cylinder is grounded so the total charge on the two shells is Q on the outer and $-Q$ on the inner shell. The electric field for $r > b$ and for $r < a$ is zero by Gauss's theorem. The electric field for $a < r < b$ is found from Gauss's law,

$$E4\pi r^2 = -Q/\epsilon_0 \quad \text{so that} \quad E(r) = -\frac{Q}{4\pi\epsilon_0 r^2} \tag{15}$$

Now we find the potential difference by integration,

$$V_b = -\int_a^b \vec{E}d\vec{l} = \int_a^b \frac{Q}{4\pi\epsilon_0 r^2}dr = -\frac{Q}{4\pi\epsilon_0}[\frac{1}{b} - \frac{1}{a}] = \frac{Q(b-a)}{4\pi ab\epsilon_0} \tag{16}$$

Again $V_a = 0$ as the inner cylinder is grounded. We then have,

$$Q = \frac{4\pi ab\epsilon_0}{b-a}V \tag{17}$$

so that $C_{sphere} = 4\pi ab\epsilon_0/(b-a)$. Even an isolated sphere carrying charge Q can be considered to have a capacitance, that is found by taking the limit $b \to \infty$, yielding $C_{isolated} = 4\pi a\epsilon_0$. Note that we don't keep the negative sign, as the capacitance is a positive quantity.

For a general geometry, determination of the capacitance requires that we calculated the potential difference between two pieces of metal for a given charge on the metals or equivalently the charge on the two pieces of metal given the potential difference. The latter is easier as the voltage on all parts of the metal is the same and hence we know the voltage boundary conditions for Laplace's equation. Nevertheless in general this problem can only be solved numerically.

Energy stored in a capacitor

Since we now know that the energy is stored in the electric field we can find the energy by integration, ie $U = (\epsilon_0/2)\int u^2(\vec{r})d\vec{r}$ For example for a parallel plate capacitor, the magnitude of the electric field between the plates is $E = V/d$, therefore the energy is simply

$$U = \frac{\epsilon_0}{2}\int u^2(\vec{r})d\vec{r} = \frac{\epsilon_0}{2}(\frac{V}{d})^2 Ad = \frac{1}{2}CV^2 \tag{18}$$

This calculation can be carried out for each case above to show that this is the general result. The general result can also be found by using,

$$U = \frac{1}{2}\int \rho(\vec{r})V(\vec{r}) \to \frac{1}{2}QV = \frac{1}{2}CV^2 \tag{19}$$