PHY481 - Lecture 9: Boundary conditions and forces at charged surfaces

Griffiths: Chapter 2

Energy stored in a capacitor

Since we now know that the energy is stored in the electric field we can find the energy by integration, ie $U = (\epsilon_0/2)\int u^2(\vec{r})d\vec{r}$ For example for a parallel plate capacitor, the magnitude of the electric field between the plates is $E = V/d$, therefore the energy is simply

$$U = \frac{\epsilon_0}{2}\int u^2(\vec{r})d\vec{r} = \frac{\epsilon_0}{2}(\frac{V}{d})^2 Ad = \frac{1}{2}CV^2 \quad (1)$$

This calculation can be carried out for each case above to show that this is the general result. The general result can also be found by using,

$$U = \frac{1}{2}\int \rho(\vec{r})V(\vec{r}) \to \frac{1}{2}QV = \frac{1}{2}CV^2 \quad (2)$$

Electric field near a conducting surface

The electric field is normal to the surface of a conductor, and using Gauss's law the magnitude of this field is found to be $E_n = \sigma/\epsilon_0$.

Electric fields near two charged metal sheets

Consider two flat metal sheets with area A, of thickness t, and separated by distance d. The sheets are parallel to each other and the size of the sheets is much larger than d or a so we can neglect fringe fields. The top sheet has total charge Q_u, while the bottom sheet has total charge Q_l. We want to find the electric field above, below and between the sheets as well as the surface charge on the top and bottom surfaces of the metal sheets.

By symmetry the electric fields above and below the sheets are related by, $\vec{E}_{below} = E_{below}\hat{z} = -\vec{E}_{above}$. Using a Gaussian surface enclosing both sheets, we have,

$$2AE_{above} = \frac{Q_u + Q_l}{\epsilon_0}, \quad \text{so} \quad E_{above} = \frac{Q_u + Q_l}{2A\epsilon_0} \quad (3)$$

We can also use a Gaussian surface enclosing only the top surface of the top sheet, in that case we have

$$AE_{above} = \frac{Q_u^{top}}{\epsilon_0}, \quad \text{so} \quad Q_u^{top} = \frac{1}{2}(Q_u + Q_l) \quad (4)$$

Since the upper sheet has total charge $Q_u = Q_u^{top} + Q_u^{bottom}$, we find $Q_u^{bottom} = (Q_u - Q_l)/2$. Carrying out the same procedure for the lower sheet yields,

$$Q_l^{top} = \frac{1}{2}(Q_l - Q_u); \quad Q_l^{bottom} = \frac{1}{2}(Q_u + Q_l) \quad (5)$$

Finally the electric field in the between the sheets is found using a Gaussian surface through the center of either sheet (I choose the upper one) and passing between the sheets, to find,

$$-AE_{between} = \frac{1}{2\epsilon_0}(Q_u - Q_l); \quad \text{so} \quad E_{between} = \frac{Q_l - Q_u}{2A\epsilon_0} \quad (6)$$

Now lets check these results for the case of a parallel plate capacitor, where $Q_u = Q$ and $Q_l = -Q$, so that $E_{above} = E_{below} = Q_u^{top} = Q_l^{bottom} = 0$, and the electric field between the plates is $-Q/(A\epsilon_0)\hat{z}$

The electric field at a surface

An example of this sort is problem 2.38. We want to calculate the force on part of a conducting surface due to the charges on the other parts of the surface. Of course we can use $\vec{F} = dQ\vec{E}_{other}$, however we have to figure out the correct value for $\vec{E}_{other}$. The answer is that we must average the electric field above and below the surface, so $\vec{E}_{other} = \frac{1}{2}(E_{above} + E_{below})$. If dQ is small, the electric field used is simply that of the whole object.

To understand why the average is correct consider a sheet of charge with charge density $\sigma(\vec{r})$. Now consider the total electric field at position $\vec{r}$ as the sum of the contribution due to a small element or "patch" of charge at that position and the contributions from all other parts of the surface, labeled "other", ie, $\vec{E} = \vec{E}_{patch} + \vec{E}_{other}$. Since

$\vec{E}_{other}$ has no contribution from the surface at location $\vec{r}$, it is non-singular there. The patch itself is small and has charge density $\sigma(\vec{r})$, so its electric field is simply $\sigma/(2\epsilon_0)$ (above the surface) and $-\sigma/(2\epsilon_0)$ below the surface, we then have,

$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\epsilon_0}; \quad \vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\epsilon_0} \tag{7}$$

Solving yields,

$$\vec{E}_{other} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below}) \tag{8}$$

This argument also applies to a conducting surface though it is somewhat counterintuitive as we must remove the thin charged layer at the surface of the conductor and treat it as a sheet of charge, as above. With this construction the argument proceeds as above.

Apply this result to charged conducting surface, yields,

$$E_{other} = \frac{\sigma}{2\epsilon_0}\hat{n}; \quad \textit{so that} \quad \vec{f} = \sigma\vec{E}_{other} = \frac{\sigma^2}{2\epsilon_0}\hat{n} \tag{9}$$

where $\vec{f}$ is the force per unit area so that its magnitude is the pressure, i.e. $P = |\vec{f}| = \epsilon_0 E^2/2$, where $\vec{E}$ is the electric field just above the surface. Lets see how big this is for the earth's surface where $E \approx 110V/m$, in that case $P \approx 5 \times 10^{-8}Pa$. For comparison, atmospheric pressure is about $101kPa$ which is about 10^{13} times larger so we don't have to worry about the electrostatic pressure blowing material off the surface of the earth!

Electrostatic boundary conditions at surfaces

The boundary conditions at surfaces are clearly interesting and important. We will need these boundary conditions in order to solve the boundary condition problems related to Laplace's equation that we will study in Chapter 3. It is quite easy to find the general boundary conditions at surfaces using the integral forms of Gauss's law, Faraday's law and the definition of the potential. First consider Gauss's law for a small patch of surface, with area $\vec{da} = da\hat{n}$. If the patch is small enough, we can treat the electric field and the charge density as constant over the patch, so that,

$$\oint \vec{E} \cdot \vec{da} = (E^{\perp}_{above} - E^{\perp}_{below})da = \frac{\sigma da}{\epsilon_0} \quad \text{so} \quad E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{10}$$

Applying Faraday's law (with the $\partial\phi_B/\partial t = 0$, we have,

$$\oint \vec{E} \cdot \vec{dl} = (E^{||}_{above} - E^{||}_{below})dl = 0 \quad \text{so} \quad E^{||}_{above} - E^{||}_{below} = 0. \tag{11}$$

Note that the perpendicular parts of the contour sum to zero as the two sides are in opposite directions but have the same electric field. Finally, using $V_b - V_a = -\int_{-a}^{a} \vec{E} \cdot \vec{dl}$, with a path in the $\hat{n}$ direction, we find,

$$V_{above} - V_{below} = -\frac{a}{2}(E^{\perp}_{above} - E^{\perp}_{below}) \rightarrow 0, \quad \text{because} \quad a \rightarrow 0 \tag{12}$$

In words, the electrostatic potential and parallel component of the electric field are continuous across surfaces, while the perpendicular component has a jump discontinuity proportional to the surface charge density.