PHY481 - Midterm I (2009)

Time allowed 50 minutes. Do all questions - to get full credit you must show your working.

The general solutions to Laplace's equation with two co-ordinates allowed to vary are:
$V(x,y) = (a+bx)(c+dy) + \sum_k [A(k)cos(kx) + B(k)sin(kx)][C(k)cosh(ky) + D(k)sinh(ky)]$ (Cartesian);
$V(s,\phi) = (A + Bln(s)) + \sum_{n=1}^{\infty}(A_n s^n + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi))$ (Cylindrical);
$V(r,\theta) = \sum_{l=0}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta)$ (Spherical polar).

Problem 1. Write down Gauss's law in integral form and derive the differential form of Gauss's law from it. Write down Faraday's law in integral form and derive the differential form from it.

Solution

Gauss's law in integral form is,

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0} \tag{1}$$

Using the divergence theorem and writing, $q = \int \rho(\vec{r})d\vec{r}$, we find,

$$\int (\vec{\nabla} \cdot \vec{E}) d\vec{r} = \frac{1}{\epsilon_0} \int \rho(\vec{r}) d\vec{r} \tag{2}$$

This is satisfied if $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, the differential form of Gauss's law. Faraday's law is,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t} \tag{3}$$

Using Stokes theorem and writing $\phi_B = \int \vec{B} \cdot d\vec{a}$, we find,

$$\int \vec{\nabla} \wedge \vec{E} \cdot d\vec{a} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{a} \tag{4}$$

This is satisfied if $\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, the differential form of Faraday's law.

Problem 2. A charge, q at position $\vec{r}' = (0,0,d)$ is above a conducting half space that lies in the region $z < 0$ so that its surface is at $z = 0$. Using the image charge method in Cartesian co-ordinates, write down the potential in the region $z > 0$ and show that the electric field is normal to the surface of the conductor. Find an expression for the induced charge at the surface of the conductor (do not integrate it) and find the work required to bring the charge from ∞ to the position $\vec{r}'$.

Solution

The solution is constructed by noting that we can satisfy the boundary condition $V(x,y,0) = 0$ with the function,

$$V(\vec{r}) = k[\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}]. \tag{5}$$

Using $\vec{E} = -\vec{\nabla}V$, we find,

$$E_x = -\frac{\partial V}{\partial x} = kq[\frac{x}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x}{(x^2+y^2+(z+d)^2)^{3/2}}] \tag{6}$$

This reduces to zero at $z = 0$ as required. A similar expression is found for E_y, while the electric field in the z-direction is,

$$E_z = -\frac{\partial V}{\partial z} = kq[\frac{z-d}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{z+d}{(x^2+y^2+(z+d)^2)^{3/2}}] \tag{7}$$

In the limit $z \to 0$, this gives $E_z(z=0) = -\frac{2kqd}{(x^2+y^2+d^2)^{3/2}}$.

The "screening" or induced charge density at the surface of the conductor is $\sigma(x,y,0) = \epsilon_0 E_z(x,y,0)$ which yields,

$$\sigma(x,y,0) = \frac{-dq}{2\pi(x^2+y^2+d^2)^{3/2}} \tag{8}$$

where $k = 1/(4\pi\epsilon_0)$ was used.

The force on the charge q,

$$\vec{F_q} = q\vec{E}_{-q}(x=0, y=0, d) = \frac{-kq^2}{4d^2}\hat{z}. \tag{9}$$

This is equal to $kqq'/(2d)^2\hat{z}$ as it must. The work is

$$W = -\int_{\infty}^{d} \frac{-kq^2}{4z^2}dz = \frac{-kq^2}{4d} \tag{10}$$

This is one half the naive (incorrect) energy, $-kq^2/2d$, because the electric field is finite only in the upper half plane.

Problem 3. A grounded conducting cylinder of radius R has its axis along the $\hat{z}$ direction and is in a uniform electric field $\vec{E} = E_0\hat{x}$. Find an expression for the potential for $s > R$.

Solution

The electric field produces potential $-E_0 x = -E_0 r cos\phi$. To satisfy this boundary condition, we need the $n = 1$ term in the solution to Laplace's equation in cylindrical co-ordinates. We can therefore first see of only the $n = 1$ term is needed so that,

$$V(s,\phi) = (A_1 r + \frac{B_1}{r})cos\phi \tag{11}$$

We can't have the $sin\phi$ term because it is not consistent with the applied field. To satisfy the applied field boundary condition, we set $A_1 = -E_0$. To ensure that the potential is zero at the surface of the grounded cylinder, we set $V(R,\phi) = 0$, which implies that $B_1 = R^2 E_0$. The solution is then,

$$V(s,\phi) = -E_0 cos\phi(r - \frac{R^2}{r}) \tag{12}$$

We have to check that this has the property that $E_{||}(R,\phi) = 0$. We find

$$E_{||}(R,\phi) = -\frac{1}{s}\frac{\partial V}{\partial \phi}|_{s=R} = -E_0 sin\phi(R - \frac{R^2}{R}) = 0 \tag{13}$$

Problem 4. Using the multipole expansion,

$$\frac{1}{|\vec{r} - \vec{r'}|} = \sum_{l=0} \frac{(r')^l}{r^{l+1}} P_l(cos\theta) \tag{14}$$

Find an expression for the electrostatic potential of a dipole that has dipole moment $\vec{p} = qd\hat{z}$ and is centered at the origin. From your expression for the potential, find the electric field of the dipole.

Solution

For a dipole charge configuration, we have,

$$V(r,\theta) = \frac{kq}{r}\sum_{l}(\frac{d}{2r})^l P_l(cos\theta) - \frac{kq}{r}\sum_{l=0}(\frac{-d}{2r})^l P_l(cos\theta) \tag{15}$$

The even terms in the sum cancel, while the odd terms add so that,

$$V(r,\theta) = \frac{kq}{r}\sum_{l\ odd} 2(\frac{d}{2r})^l P_1(cos\theta) = \frac{kq}{r}[\frac{d}{r}cos\theta + 2(\frac{d}{2r})^3 P_3(cos\theta) + ...]. \tag{16}$$

The leading term is the dipole potential $\frac{kqdcos\theta}{r^2}$. The electric field is found using,

$$\vec{E} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} = \frac{2kpcos\theta}{r^3}\hat{r} + \frac{kpsin\theta}{r^3}\hat{\theta} \tag{17}$$

Using $\hat{z} = \hat{p} = cos\theta\hat{r} - sin\theta\hat{\theta}$ this may be written as,

$$\vec{E} = k\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3} \tag{18}$$

Problem 5. Consider two concentric spherical shells. The inner shell is at potential V_a and has radius a, while the outer shell is at potential V_b and has radius $b > a$. Find the electrostatic potential for $r < a$, $a < r < b$, and $r > b$.

Solution

Since the system is spherically symmetric we can use the solution to Laplace's equation with $l = 0$, $A_0 + B_0/r$. We have to find different values for the constants in the three regions. In the first $r < a$ there is no charge so the potential cannot diverge, we therefore set $B_0(r < a) = 0$. We must also satisfy the boundary condition at $r = a$, therefore $A_0(r < a) = V_a$. The solution for $r < a$ is then $V = V_a$. In the region $a < r < b$ we have the two boundary conditions,

$$V_a = A_0 + B_0/a; \quad V_b = A_0 + B_0/b \tag{19}$$

Subtracting these two equations and solving for B_0 we find that,

$$B_0 = \frac{V_a - V_b}{(\frac{1}{a} - \frac{1}{b})} \tag{20}$$

and hence that,

$$A_0 = V_a - \frac{1}{a}\frac{V_a - V_b}{(\frac{1}{a} - \frac{1}{b})} \tag{21}$$

so the solution in this region is,

$$V(a < r < b) = V_a + (\frac{1}{r} - \frac{1}{a})\frac{V_a - V_b}{(\frac{1}{a} - \frac{1}{b})} \tag{22}$$

For $r > b$ we only have the B_0/r as the potential goes to zero at large r. With $B_0/b = V_b$, we find that $V(r > b) = V_b b/r$.

Problem 6. Consider two concentric conducting cylindrical shells. The inner shell has charge $-\lambda$ per unit length, while the outer shell has charge λ per unit length. Find the electrostatic potential difference between the shells and hence find the capacitance per unit length of the system. Find an expression for the energy per unit length stored in the capacitor.

Solution

The electric field between the shells is $-\lambda ln(s)/(2\pi\epsilon_0)$ and the potential between the cylinders is $\lambda ln(b/a)/(2\pi\epsilon_0)$. The capacitance is defined through $Q = CV$ so the capacitance per unit length is defined through $\lambda = CV/l$. From the expression for the potential difference we then have,

$$\frac{C}{l} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0}ln(b/a)} = \frac{2\pi\epsilon_0}{ln(b/a)} \tag{23}$$

The energy in a capacitor is $CV^2/2$ so the energy per unit length $CV^2/2l$ so that,

$$\frac{U}{l} = \frac{1}{2}\frac{2\pi\epsilon_0}{ln(b/a)}(\frac{\lambda ln(b/a)}{2\pi\epsilon_0})^2 = \frac{\lambda^2}{2}\frac{ln(b/a)}{2\pi\epsilon_0} \tag{24}$$