PHY481 - Midterm IB (2009)

Time allowed 50 minutes. Do all questions - to get full credit you must show your working.

The general solutions to Laplace's equation with two co-ordinates allowed to vary are:
$V(x,y) = (a+bx)(c+dy) + \sum_k [A(k)cos(kx) + B(k)sin(kx)][C(k)cosh(ky) + D(k)sinh(ky)]$ (Cartesian);
$V(s,\phi) = (A + Bln(s)) + \sum_{n=1}^{\infty}(A_n s^n + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi))$ (Cylindrical);
$V(r,\theta) = \sum_{l=0}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta)$ (Spherical polar).

Problem 1. Write down Gauss's law in integral form and derive the differential form of Gauss's law from it. Write down Faraday's law in integral form and derive the differential form from it. Derive the boundary conditions

$$E^{\|}_{above} - E^{\|}_{below} = 0; \quad E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{1}$$

for the parallel electric field and perpendicular electric field just above and just below a thin surface that has charge density per unit area σ.

Solution

Gauss's law in integral form is,

$$\oint \vec{E}\cdot d\vec{a} = \frac{q}{\epsilon_0} \tag{2}$$

Using the divergence theorem and writing, $q = \int \rho(\vec{r})d\vec{r}$, we find,

$$\int (\vec{\nabla}\cdot\vec{E})d\vec{r} = \frac{1}{\epsilon_0}\int \rho(\vec{r})d\vec{r} \tag{3}$$

This is satisfied if $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, the differential form of Gauss's law. Faraday's law is,

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial\phi_B}{\partial t} \tag{4}$$

Using Stokes theorem and writing $\phi_B = \int \vec{B}\cdot d\vec{a}$, we find,

$$\int \vec{\nabla}\wedge\vec{E}\cdot d\vec{a} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{a} \tag{5}$$

This is satisfied if $\vec{\nabla}\wedge\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, the differential form of Faraday's law.

Choose a small rectangular contour with the surface lying within the contour and two sides of the rectangle parallel and two sides perpendicular to the surface. Applying the integral form of Faraday's law (with the $\partial\phi_B/\partial t = 0$), we have,

$$\oint \vec{E}\cdot\vec{dl} = (E^{\|}_{above} - E^{\|}_{below})dl = 0 \quad \text{so} \quad E^{\|}_{above} - E^{\|}_{below} = 0. \tag{6}$$

Note that the perpendicular parts of the contour sum to zero as the two sides are in opposite directions but have the same electric field.

Now apply Gauss's law to a small cubic Gaussian surface where the surfaces of the cube are either parallel or perpendicular to the surface, and the surface lies within the cube. If the cube is small enough, we can treat the electric field and the charge density as constant over the patch, so that,

$$\oint \vec{E}\cdot\vec{da} = (E^{\perp}_{above} - E^{\perp}_{below})da = \frac{\sigma da}{\epsilon_0} \quad \text{so} \quad E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0} \tag{7}$$

The parallel components cancel so we are left with the perpendicular components.

Problem 2. A thin disc of uniform charge density σ and radius R is centered at the origin with its normal along the $\hat{z}$ axis. Find the potential of the disc on the z-axis and show that it reduces to a point charge form for large z and to the form for an infinite sheet of charge as $z \to 0$.

Solution

Using the superposition formula, the potential on the z-axis is given by,

$$V(z) = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{sdsd\phi}{(s^2+z^2)^{1/2}} \tag{8}$$

Doing the integrals yields,

$$V(z) = \frac{\sigma}{4\pi\epsilon_0}2\pi(s^2+z^2)^{1/2}|_0^R = \frac{\sigma}{2\epsilon_0}[(R^2+z^2)^{1/2}-z] \tag{9}$$

As $z \to 0$, we use an expansion in z^2/R^2, so that to leading order,

$$V(z) = \frac{\sigma}{2\epsilon_0}[(R^2+z^2)^{1/2}-z] \to \frac{\sigma}{2\epsilon_0}[R-z] \to C - \frac{\sigma}{2\epsilon_0}z, \tag{10}$$

which is the potential corresponding to a contant electric field of $\sigma/2\epsilon_0$. As $z \to \infty$, we can use an expansion in R^2/z^2, so that to leading order,

$$V(z) = \frac{\sigma}{2\epsilon_0}[z(1+\frac{R^2}{2z^2})-z] \to \frac{\sigma}{2\epsilon_0}\frac{R^2}{2z} \to \frac{kQ}{z} \tag{11}$$

where we used $Q = \pi R^2\sigma$ and $k = 1/(4\pi\epsilon_0)$. At long distances the potential looks like that of a point charge, as expected.

Problem 3. A grounded conducting sphere of radius R is centered at the origin and is in a uniform electric field $\vec{E} = E_0\hat{z}$. Find an expression for the potential outside the sphere, i.e. for $r > R$. Find an expression for the induced charge density on the surface of the sphere (it should be a function of θ). If the sphere is now disconnected from ground and there is an additional charge Q placed on the sphere, what is the new expression for the potential?

Solution

The potential corresponding to a uniform electric field in the $\hat{z}$ direction is $-E_0z = -E_0rcos\theta$. This has the angular form corresponding to the $(l = 1)$ term in the solution to Laplace's equation in spherical polar co-ordinates. If there is no net charge on the cylinder and the cylinder is grounded, we set the $l = 0$ term to zero. Therefore we try the solution,

$$V(r,\theta) = (A_1r + \frac{B_1}{r^2})cos\theta \tag{12}$$

To satisfy the boundary condition at infinity, we set $A_1 = -E_0$. To ensure that the potential goes to zero at the surface of the sphere, we set $-E_0R + B_1/R^2 = 0$, so that, $B_1 = E_0R^3$. The solution is then,

$$V(r,\theta) = -E_0(r - \frac{R^3}{r^2})cos\theta \tag{13}$$

It is clear that the parallel electric field at the surface of the conducting sphere, i.e. $-\frac{1}{r}\frac{\partial V}{\partial\theta}|_R = 0$ as required. The induced charge density at the surface is found from

$$\sigma = \epsilon_0E_n(R) = -\epsilon_0\frac{\partial V}{\partial r}|_R = 3\epsilon_0E_0cos\theta \tag{14}$$

If the sphere is disconnected from ground and a charge Q is added to the conducting sphere, then the potential outside the sphere is given by,

$$V(r,\theta) = -E_0(r - \frac{R^3}{r^2})cos\theta + \frac{kQ}{r} + A_0 \tag{15}$$

where A_0 is an arbitrary reference potential, that we can choose to be zero.

Problem 4. Three isolated infinite metal (i.e. conducting) slabs have their normals along the z axis, so the slabs are parallel to each other. The top slab has total charge Q, while the bottom slab has total charge q. Find the charge on the top surfaces of each slab and on the bottom surfaces of each slab.

Solution

Label the surface charges, starting from the top of the top slab, $q_1, q_2, q_3, q_4, q_5, q_6$. We have,

$$q_1 + q_2 = Q; \quad q_3 + q_4 = 0; \quad q_5 + q_6 = q \tag{16}$$

By superposition, the electric field above the top slab is $E_1 = \frac{q+Q}{2\epsilon_0 A}$, while the electric field below the bottom slab is $E_4 = \frac{-(q+Q)}{2\epsilon_0 A}$, where A is the area of a slab. When using superposition we add up the contributions of all six surface charges. Now we use Gauss's law, by taking a cubic Gaussian surface, with surfaces of area A with faces parallel and perpendicular to the slabs. Choose one face to lie within the top slab with another face above the top slab, Then we have, $E_1 = \sigma_1/\epsilon_0 = q_1\epsilon_0 A$, therefore, $q_1 = \frac{q+Q}{2}$. Similarly $q_6 = \frac{q+Q/}{2}$. By charge conservation we then have $q_2 = \frac{Q-q/}{2}$ and $q_5 = \frac{q-Q}{2}$. Taking a cubic Gaussian surface with one face inside the top slab and another inside the middle slab, it is evident that the total flux is zero through the Gaussian surface is zero. Therefore the enclosed charge is zero so we must have $q_2 + q_3 = 0$, so $q_3 = -q_2 = \frac{q-Q}{2}$. Charge conservation then gives $q_4 = -q_3 = \frac{Q-q}{2}$

Problem 5. An infinitely long cylindrical shell of radius R has charge density distribution $\sigma = \sigma_0 sin(2\phi)$ on its surface, where σ_0 is a constant. Find the electric field inside the cylinder.

Solution

The charge density at the surface is related to the perpendicular component of the electric field (see Eq. (1)), so the angular dependence of the potential must be the same as the angular dependence of the charge density boundary condition, we therefore choose, the $n = 2$ term in the general solution,

$$V(s,\phi) = (A_2 s^2 + \frac{B_2}{s^2}) sin2\phi \tag{17}$$

where we set the constant $C_2 = 1$ without loss of generality. In addition, we need to ensure that the solution is non-singular for $s \to 0$ as there are no charges in the interior of the cylinder, we therefore set $B_2 = 0$ for $s < R$. To find the relation between the charge density and the potential, we need to use the boundary contitions given in Eq. (1). We can also use the fact that the potential is continuous across the cylindrical surface. The solutions inside and outside the cylindrical shell are,

$$V_{out} = \frac{B_2}{s^2}) sin2\phi; \quad V_{in} = A_2 s^2 sin2\phi \tag{18}$$

Continuiting of the potential at the surface $V_{out}(R) = V_{in}(R)$ implies that $B_2/R^2 = A_2 R^2$. The equation for the perpendicular component of the electric field is then,

$$E^{\perp}_{out}(R) - E^{perp}_{in}(R) = -\frac{\partial V_{out}}{\partial r}|_R + \frac{\partial V_{in}}{\partial r}|_R = \frac{\sigma}{\epsilon_0} = \frac{\sigma_0 sin(2\phi)}{\epsilon_0} \tag{19}$$

Doing the derivatives and using $B_2 = A_2 R^4$ gives,

$$2A_2 R sin(2\phi) + 2A_2 R sin(2\phi) = \frac{\sigma_0 sin(2\phi)}{\epsilon_0} \tag{20}$$

so that $A_2 = \frac{\sigma_0}{4\epsilon_0 R}$. The potential inside the cylinder is,

$$V(s < R, \phi) = V_{in} = \frac{\sigma_0}{4\epsilon_0 R} s^2 sin2\phi \tag{21}$$

The electric field inside the cylinder is,

$$\vec{E}(s < R, \phi) = -\frac{\partial V}{\partial s}\hat{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{\phi} = -\frac{\sigma_0}{2\epsilon_0 R} s[sin(2\phi)\hat{s} + cos(2\phi)\hat{\phi}] \tag{22}$$