PHY481 - Midterm II (2009)

Time allowed 50 minutes. Do all questions - to get full credit you must show your working.

Problem 1. a) Write down the integral and differential forms of Maxwell's equations. b) Set the source terms in the differential forms to zero and from the resulting equations derive the wave equations for the electric and magnetic fields. c) Assume the electric field solution to the wave equation is $\vec{E} = E_0\hat{x}Cos(kz - \omega t + \delta)$ and hence show that $c = 1/(\mu_0\epsilon_0)^{1/2}$.

Solution

a) *Maxwell's equations are:*

$$\phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}; \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\phi_B = \oint \vec{B} \cdot d\vec{a} = 0; \qquad \vec{\nabla} \cdot \vec{B} = 0 \tag{2}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0\epsilon_0 \frac{d\phi_E}{dt} \qquad \vec{\nabla} \wedge \vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \tag{3}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\phi_B}{dt}; \qquad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4}$$

b) *Setting the source terms to zero in the differential forms gives,*

$$\vec{\nabla} \cdot \vec{E} = 0; \quad \vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \vec{\nabla} \wedge \vec{B} = \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}; \tag{5}$$

Taking a time derivative of the third of these equations and using the last equation yields,

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{so that} \quad \nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2 \vec{E}}{\partial t^2} \tag{6}$$

where we used the $\vec{\nabla} \cdot \vec{E} = 0$ to find the final result. In a similar way if we take a time derivative of Ampere's law in free space and use Faraday's law we get,

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{so that} \quad \nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2 \vec{B}}{\partial t^2} \tag{7}$$

c) *Substituting* $\vec{E} = E_0\hat{x}Cos(kz - \omega t + \delta)$ into the wave equation for the electric field gives $k^2 = \mu_0\epsilon_0\omega^2$. Since we also have $\omega = vk$ (or $v = f\lambda$), where v is the wave speed, we find that the speed of EM waves is $v = c = 1/(\mu_0\epsilon_0)^{1/2}$.

Problem 2. An infinite solenoid with circular cross-section has n turns per unit length and radius R. It carries DC current I. a) Use Ampere's law to find the magnetic field inside and outside the solenoid. b) Use your result to find the self-inductance of the solenoid. c) Use the relation $\phi_B = \oint \vec{A} \cdot d\vec{l}$ to find the vector potential inside and outside the solenoid.

Solution

a) Take a rectangular contour outside the solenoid everywhere and at infinity on one side, and close to the solenoid on the other side. These sides of the contour are parallel with the solenoid axis. The enclosed current is zero while the magnetic field is zero at infinity. The two sides of the contour that are perpendicular to the axis of the solenoid have equal an opposite contributions. Therefore the side parallel to the solenoid axis and close to the solenoid also make zero contribution. Therefore the mangetic field outside the solenoid is zero. Now taking a rectangular contour with one side inside the contour and parallel to the axis of the solenoid and the other side parallel and outside the solenoid. The sides perpendicular to the solenoid axis make equal and opposite contributions and hence cancel in the contour integral. There is no magnetic field outside the solenoid and the direction of the magnetic field inside the solenoid is in the direction of the axis of the solenoid and given by the right hand rule. We then have,

$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 Ni \quad \text{so that} \quad B_{inside} = \mu_0 \frac{N}{l} i = \mu_0 ni \tag{8}$$

b) The self-inductance is defined through $Li = N\phi = N\pi R^2 B = \mu_0 \frac{N^2}{l} i\pi R^2$, so that $L = \mu_0 N^2 \pi R^2/l$

c) To find the vector potential from $\oint \vec{A}\cdot d\vec{l} = \phi_B$, we choose a circular contour concentric with the solenoid and of radius s. The vector potential is in the direction of the current, ie. $\hat{\phi}$. The flux enclosed by the circular contour is $\phi_B(s > R) = \pi R^2 \mu_0 ni$, $\phi(s < R) = \pi s^2 \mu_0 ni$. The contour integral gives, $A(s)2\pi s$, so the vector potential is

$$\vec{A}(s > R) = \frac{\pi R^2 \mu_0 ni}{2\pi s}\hat{\phi} = \frac{R^2 \mu_0 ni}{2s}\hat{\phi}; \quad \vec{A}(s < R) = \frac{s\mu_0 ni}{2}\hat{\phi} \tag{9}$$

Problem 3.

An infinite straight wire carries current I in the $\hat{z}$. A square loop, of side "a", is placed with its closest side parallel to the infinite wire and at distance s from the infinite wire. a) Find the flux passing through the square loop. b) If the loop now moves with speed v in the $\hat{z}$ direction what is the emf induced in the loop? c) If the loop rotates about the infinite wire with an increasing angular speed $\omega = \alpha t$, but at fixed radial location and with its normal in the $\hat{\phi}$ direction, what is the induced emf in the loop? d) If the loop is pulled away from the wire in the $\hat{s}$ direction with speed v, what is the induced emf?

Solution

a) The flux through the loop is given by,

$$\phi_B = a\int_s^{s+a} \frac{\mu_0 i}{2\pi s'} ds = \frac{\mu_0 ia}{2\pi}[ln(a+s) - ln(s)] \tag{10}$$

b) zero, because ϕ_B is constant.
c) zero, because ϕ_B is constant.
d) From Faraday's law, we have,

$$emf = -\frac{d\phi_B}{dt} = -\frac{\mu_0 ia}{2\pi}[\frac{1}{a+s} - \frac{1}{s}]\frac{ds}{dt} = \frac{\mu_0 ia^2 v}{2\pi s(s+a)} \tag{11}$$

where $ds/dt = v$.

Problem 4. A thin disc with charge density σ and radius R has its normal along $\hat{z}$ and has constant angular velocity $\omega\hat{z}$. Find an expression for the magnetic field on the z-axis for $z > 0$. (A bonus point if you can answer this question- Show that result takes a dipole form at long distances?)

Solution

The current density is given by $K = \frac{1}{length}\frac{dq}{dt} = \frac{1}{ds}\frac{\sigma s ds d\phi}{dt} = \sigma s\omega$. The direction of K is $\hat{\phi}$. For the ring of current at radius s, we have,

$$dB_z(s) = \frac{\mu_0}{4\pi}\int_0^{2\pi} \sigma s\omega s d\phi \frac{cos(\theta(s))}{r^2(s)} = \frac{\mu_0}{2}\sigma s\omega s d\phi \frac{s}{(s^2+z^2)^{3/2}} \tag{12}$$

where we used $r(s) = (s^2+z^2)^{1/2}$, $cos(\theta(s)) = s/r(s)$. The field on the z axis is then,

$$B_z(s) = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{s^3 ds}{(s^2+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{2}[\frac{s^2+2z^2}{(s^2+z^2)^{1/2}}]|_0^R = \frac{\mu_0\sigma\omega}{2}[\frac{R^2+2z^2}{(R^2+z^2)^{1/2}} - 2z] \tag{13}$$

Bonus question - A systematic expansion to second order is required as the zeroth and first order terms cancel, so we have,

$$B_z(s) = \frac{\mu_0\sigma\omega}{2}[2z^2(1+\frac{R^2}{2z^2})\frac{1}{z}(1 - \frac{1}{2}\frac{R^2}{z^2} + \frac{3}{8}(\frac{R^2}{z^2})^2) - 2z] = \frac{\mu_0\sigma\omega}{2}\frac{R^4}{4z^3} \tag{14}$$

This is the same as the field along the normal of a current ring, with dipole moment $m = \frac{\pi}{4}\sigma\omega R^4$. Alternatively, use the formula for the field of a magnetic dipole $\vec{B} = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r}-\vec{m}}{r^3}$. Taking $\vec{m} = m\hat{z}$ and evaluating the field on the $\hat{z}$ axis where $\vec{r} = z\hat{z}$, this reduces to $\frac{\mu_0}{4\pi}\frac{2m}{z^3}\hat{z}$. now compare this with Eq. (14) to find m. This may also be found by noting that the disc is made up of rings, and each ring has a dipole moment $m(s) = di(s)a(s)$, so the dipole moment of the disc is $m = \int_0^R a(s)K ds = \int_0^R \pi s^2 \sigma s\omega ds = \pi\sigma\omega R^4/4$.