PHY481 - Review sheet for Midterm 1

Griffiths: Chapters 1-3

Electric Field

There are two types of charge and they interact through Coulomb's law $\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r^3}\vec{r}$. The interaction between many charges is found by using superposition. The electric field due to a charge Q through, $\vec{E} = \frac{\vec{F}}{q}$, so that the electric field is the force per unit charge. Since the unit of force is the Newton (N), the unit of electric field is N/C, where C is the unit of electric charge, the Coulomb. The electric field at position $\vec{r}$ due to a point charge at position $\vec{r}'$ is $\vec{E}(\vec{r},\vec{r}') = k\frac{Q}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$ The superposition principle indicates that the electric field at position $\vec{r}$ due to n charges at positions $\vec{r}_1...\vec{r}_n$ is given by the vector sum,

$$\vec{E}(\vec{r}) = \sum_{i=1}^{n} k\frac{Q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i) = k\int \frac{d\vec{r}'\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') \tag{1}$$

where $\rho(\vec{r}') = \sum_i Q_i\delta(\vec{r}'-\vec{r}_i)$. When treating continuous charge distributions, we may be given a charge per unit length, λ, a charge per unit area σ or a charge per unit volume ρ. Typical superposition problems: ring of charge; disc of charge; line segment etc. Note that you can always solve the superposition problem for the potential and then take a gradient to find the electric field. This is often easier.

An electric field line is a series of vectors where at each point the vector points in the direction of the force on a unit charge at that point and it has a length equal to the magnitude of the force. ie. we plot the vector function $\vec{E}$. The properties of electric field lines constructed in this way are as follows. (i) At each point along an electric field line, the force on a positive test charge is in a direction tangent to the field line at that point. This implies that electric field lines come out of positive charges and go into negative charges. (ii) The density of lines at any point in space is proportional to the magnitude of the electric field at that point. (iii) Electric field lines begin and/or end at charges, or they continue off to infinity. i.e. they do not begin or end in free space. (iv) Electric field lines do not cross.

Conductors: If there is no current flowing, then the electric field is zero, $\vec{E} = 0$, inside a conductor, and at the surface of a conductor the electric field is normal to the surface (know the reasoning behind this).

The integral form of Gauss's law in free space is,

$$\phi_E = \oint_S \vec{E}\cdot d\vec{a} = \frac{q_{encl}}{\epsilon_0} \tag{2}$$

where $q_{encl} = \sum_{i\varepsilon\tau} q_i = \int_\tau \rho(\vec{r})d\vec{r}$ where $\rho(\vec{r})$ is the charge density. The differential form of Gauss's law: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (know how to derive this). Know how to solve Gauss's law problems in spherical, cylindrical and planar geometries, including at a conducting surface. It is good memorize some of these: $kQ\hat{r}/r^2$; $\lambda\hat{s}/(2\pi\epsilon_0 s)$; $\sigma\hat{n}/(2\epsilon_0)$; $\sigma\hat{n}/\epsilon_0$ and their derivations. Another useful result is the electric field inside a uniform sphere of charge $\rho\vec{r}/(3\epsilon_0)$.

Electrostatic potential

The difference in potential energy between two positions a and b is $\Delta U_{ab} = \int_a^b \vec{F}_{ext}\cdot d\vec{l} = -\int_a^b \vec{F}\cdot d\vec{l}$. We define $V_{ab} = U_{ab}/q$ so that

$$\Delta V_{ab} = -\int_a^b \vec{E}\cdot d\vec{l}, \tag{3}$$

for any path between a and b. This definition is consistent with the differential form, where $\vec{\nabla}\wedge\vec{E} = 0$ implies that $\vec{E} = -\vec{\nabla}V$. Since $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, we have $\nabla^2 V = -\rho/\epsilon_0$ (Poisson's equation). The special case $\rho = 0$ is Laplace's equation. The superposition formula for the potential is

$$V(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r}-\vec{r}_i|} \to \int d\vec{r}'\frac{k\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}. \tag{4}$$

Typical superposition problems: ring of charge; disc of charge; line of charge; shell of charge; uniform sphere of charge. It is good to memorize some of the results: kQ/r (outside a uniform sphere or shell of charge); $-\lambda ln(s)/(2\pi\epsilon_0) + C$ (outside a line or uniform cylinder of charge); $-E_0 z + C$ (sheet of charge or uniform electric field in z-direction). You should know how to derive these by integration of the Gauss's law results.

The potential energy of a small element of charge in a potential is $U = qV \to \int \rho(\vec{r})V(\vec{r})$ (constant V), however the energy required to set up a charge distribution is $U_n = \frac{1}{2}\sum_{i\neq j}^{n,n}\frac{kQ_iQ_j}{r_{ij}} = \sum_{i<j}^{n,n}\frac{kQ_iQ_j}{r_{ij}} \to \frac{1}{2}\int\rho(\vec{r})V(\vec{r})$. Each pair

interaction is counted once (you should know the reasoning behind this factor of (1/2)). The energy density in the electric field is $u(\vec{r}) = \frac{1}{2}\epsilon_0 \vec{E}^2$.

The capacitance is defined through a geometry consisting of two separated conductors one carrying charge Q and the other a charge $-Q$. The voltage difference between the two conductors is V. Then we define $Q = CV$. You should know how to find C for parallel plate, co-axial cylinder, and concentric sphere cases (also the case of an isolated conducting sphere where the other electrode is defined to be at infinity). It is useful to know the results for the parallel plate ($\epsilon_0 A/d$); concentric cylinder ($2\pi\epsilon_0/(ln(b/a))$); and concentric sphere ($4\pi\epsilon_0/(1/a - 1/b)$ - the isolated sphere case is when $b \to \infty$. The energy stored in a capacitor follows from $U = (1/2)\int \rho V \to QV/2 = CV^2/2$.

More advanced methods

The boundary conditions across a charged surface are (Know how to derive these):

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}; \quad E^{\parallel}_{above} - E^{\parallel}_{below} = 0; \quad V_{above} - V_{below} = 0. \tag{5}$$

Image charge method: Point charge near a grounded conducting surface $q' = -q; z' = -d$; line charge near a grounded conducting cylinder $\lambda' = -\lambda; b = R^2/a$; point charge near a grounded conducting sphere $q' = -qR/a; b = R^2/a$. Potential found by superposition and applies outside conductor. If the conductors are neutral instead of grounded, an additional charge is added to ensure neutrality. In sphere and cylinder cases the additional charge is at the center of the conductor. Image charge method also works for polygonal section taken out of a polygonal conductoring wire, provided the section is from a polygon with an even number of sides. The image charges then alternate in each remaining section of the conductor. Know how to calculate the electric field, induced charge at the surface, the force on the real charge, and the energy required to bring the charge in from infinity.

The general solutions to Laplace's equation with two co-ordinates allowed to vary are:
$V(x,y) = (a+bx)(c+dy) + \sum_k [A(k)cos(kx) + B(k)sin(kx)][C(k)cosh(ky) + D(k)sinh(ky)]$ (Cartesian);
$V(s,\phi) = (A + Bln(s)) + \sum_{n=1}^{\infty}(A_n s^n + \frac{B_n}{s^n})(C_n cos(n\phi) + D_n sin(n\phi))$ (Cylindrical);
$V(r,\theta) = \sum_{l=0}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(cos\theta)$ (Spherical polar).
Good example problems are: square with fixed potential or charge density on two sides and zero potential or charge on other two, find potential inside; cylindrical shell with fixed potential or charge density with opposite signs on top half and bottom half (find potential inside and outside); spherical shell with with fixed potential or charge density with opposite signs on top half and bottom half (find potential inside and outside). It is good to know the first three Legendre Polynomials: $P_0 = 1; P_1 = u; P_2 = (3u^2 - 1)/2$; where $u = cos\theta$.

Know how to solve the simpler problems: concentric shells or cylinders with fixed potentials (no angle dependence); grounded or charged conducting sphere or cylinder in a constant electric field (only the first angle terms).

Multipole expansion can be derived from, $\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}\frac{r'^l}{r^{l+1}}P_l(cos\theta) \quad r > r'$. For a dipole this leads to $V_{dipole} = k(\vec{p}\cdot\hat{r})/r^2$. Outside a general charge distribution $\rho(\vec{r}')$ the potential can be expanded as a multipole series, $V_Q + V_{dipole} + V_{quadropole} + ...$ The first term is the potential for a point charge, but with $Q = \int \rho(\vec{r}')d\tau$. The second term is the dipole potential where the dipole moment is given by $\vec{p} = \sum_i q_i \vec{r}_i = \int \vec{r}'\rho(\vec{r}')d\tau$.

General features of electrostatics

Only saddle points in the interior: Solutions to Laplace's equation on any domain have no minima or maxima in the interior of the domain, there can only be saddle points in the interior. All minima or maxima occur on the boundaries of the domain. *Earnshaw's theorem:* The electric field derived from a potential obeying Laplace's equation has the property that on the interior of a domain there is always at least one direction with a positive field and one direction with a negative field. Therefore charge configurations in electrostatics are unstable, with opposite charges tending to collapse and like charges moving an infinite distance appart. *Uniqueness theorem:* Two solutions to Laplace's equations, V_1 and V_2, found using the same fixed voltage (Dirichelet) boundary conditions must be the same solution (know how to prove this). Two solutions to Laplace's equation satisfy $\int \rho_1 V_2 d\vec{r} = \int \rho_2 V_1 d\vec{r}$.

Other things you should know

Dipole in a constant field $U = -\vec{p}\cdot\vec{E}$, $\vec{\tau} = \vec{p}\wedge\vec{E}$. The gradient in the three co-ordinate systems

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}), \quad \vec{\nabla} = (\frac{\partial}{\partial s}, \frac{1}{s}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}), \quad \vec{\nabla} = (\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{rsin\theta}\frac{\partial}{\partial \phi}). \tag{6}$$

Divergence theorem, Stokes theorem. Transformation of unit vectors from cylindrical to cartesian co-ordinates. Use of the gradient in spherical polars to show that the electric field of a dipole is $k(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p})/r^3$.