PHY481 - Review sheet for Midterm 2
Griffiths: Chapters 5,7 and Sections 9.2.1, 9.2.2

Static magnetic fields: Static magnetic fields are generated by DC currents or by the intrinsic magnetic moment of elementary particles. DC current in a wire is $\vec{i} = i\vec{l}$, while flow over a surface is usually denoted as current per unit length $\vec{K}$. Flow in the bulk is the current density $\vec{j}$. Fields are calculated using Ampere's law or the Biot-Savart law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{encl} \quad (Ampere); \quad d\vec{B} = \frac{\mu_0 i d\vec{l}\wedge\hat{r}}{4\pi r^2} \quad (Biot-Savart) \tag{1}$$

For symmetric systems, such as a wire, sheet of current or solenoid it is convenient to calculate magnetic fields using Ampere's law. In cases that lack symmetry, the magnetic field of a wire segment is found by superposition using the Biot-Savart law in Eq. (1). For a three dimensional current distribution the Biot-Savart law is, $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{j}(\vec{r}')\wedge(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ Typical superposition problems include: rotating ring of charge; rotating disc of charge; circular and polygonal loops etc. Useful results to know: field near a long wire oriented in the $\hat{z}$ direction is $\vec{B}(s) = \frac{\mu_0 i}{2\pi s}\hat{\phi}$; the fieldc at the center of a circular ring with normal $\hat{z}$ and radius R, is $\vec{B} = \frac{\mu_0 i}{2R}\hat{z}$; the field inside an infinite uniform solenoid is $B = \mu_0 n i$, along its axis.

The magnetic field of a magnetic dipole is like that for an electric dipole, i.e., $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{3(\vec{m}\cdot\hat{r})\hat{r}-\vec{m}}{r^3}$. At long distances a planar current ring has the magnetic field of a dipole, with $\vec{m} = i\vec{a}$, where $\vec{a}$ is the area of the ring. This result generalizes to non-planar rings, provided we define an appropriate area that is usually written $\int d\vec{a}$ and is a vector sum of the area elements making up the ring. The earth's magnetic field is quite close to a dipole form and the magnitude at the earth's surface is about $50\mu T = 0.5G$ but it varies quite considerably from place to place.

Magnetic force: A charge, q, moving at velocity $\vec{v}$ through a static magnetic field $\vec{B}$ experiences a magnetic force $\vec{F} = q\vec{v}\wedge\vec{B}$. Similarly a current carrying wire in a magnetic field experiences a force $\vec{F} = i\vec{l}\wedge\vec{B}$. If both magnetic and electric fields are present then the total force on a moving charge is the Lorentz force, $\vec{F} = q(\vec{E}+\vec{v}\wedge\vec{B})$. Typical problems are the velocity selector, cyclotrons, helical motion, DC motor. This force is also important in understanding charge motion and magnetic drag in Faraday's law problems. The force per unit length between two wires is $\mu_0 i_1 i_2/(2\pi d)$ and is attractive for currents in the same direction. A magnetic dipole in a uniform magnetic field experiences a torque $\vec{m}\wedge\vec{B}$.

Magnetic vector potential: The magnetic field is related to the magnetic potential through $\vec{B} = \vec{\nabla}\wedge\vec{A}$. The magnetic potential may be calculated by using,

$$\vec{A} = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad or \quad \oint \vec{A}\cdot d\vec{l} = \phi_B \tag{2}$$

In regions of space where there is no current it is also possible to write $\vec{B} = -\vec{\nabla}\phi_m$, where ϕ_m is a scalar potential and obeys Laplace's equation. However the vector potential is the more fundamental and generally applicable form. Using the differential form of Ampere's law, we have,

$$\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{A}) = \mu_0\vec{j} \quad \rightarrow \text{(Coulomb Gauge)} \quad \nabla^2\vec{A} = -\mu_0\vec{j} \tag{3}$$

where the Coulomb gauge is defined through $\vec{\nabla}\cdot\vec{A} = 0$. The vector magnetic potential has a lot of freedom and even in the Coulomb gauge has some freedom remaining. For example the magnetic potential that leads to magnetic field $\vec{B} = B\hat{z}$, may have the general form $\vec{A} = B(-ay, bx, f(z))$, with $a+b = 1$. The Coulomb gauge condition only ensures that $f(z) = constant$. The vector potential of a magnetic dipole is $\frac{\mu_0\vec{m}\wedge\hat{r}}{4\pi r^2}$. Contour integration can be used to find the vector potential for the solenoid, a current carrying wire and a sheet of current.

Boundary conditions in magnetostatics: In magnetostatics, the boundary conditions across a current carrying surface are (know how to derive these),

$$\vec{A}_{above} - \vec{A}_{below} = 0; \quad B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K; \quad B^{\perp}_{above} - B^{\perp}_{below} = 0. \tag{4}$$

These equations follow from the integral forms of magnetic flux, Ampere's law, and Gauss's law (in magnetostatics)

$$\phi_B = \int \vec{B}\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}; \quad \oint \vec{B}\cdot d\vec{l} = \mu_0 i; \quad \oint \vec{B}\cdot d\vec{a} = 0 \tag{5}$$

Flux linkage, self-inductance, mutual inductance: The amount of flux generated in a loop due to a current in the loop are related to each other through $\phi_B = Li$. If there are N loops then $N\phi_B = Li$ so the flux linkage increases in proportion to the number of loops. This formula is used to calculate the self-inductance, with typical examples being the solenoid and the co-axial cable. Flux may also be produced by circuits that are not corrected to our pickup loop, in that case the flux linkage is defined through $N_1\phi_1 = M_{12}i_2$, where i_2 is the current in the drive loop and ϕ_1 is the flux in the pickup loop. M_{12} is the mutual inductance. Similarly $N_2\phi_2 = M_{21}i_1$. In general $M_{ij} = M_{ji}$. Typical cases are a loop near a wire and concentric solenoids.

Current, current density, conductance: The rate of change of charge inside a small volume is related to the divergence of current through the surface, so that,$-\frac{dq}{dt} = -\frac{\partial}{\partial t}\int \rho(r)d\tau = \int \vec{j}\cdot d\vec{a}$, so that $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$, the continuity equation. In steady state we then have $\vec{\nabla}\cdot\vec{j} = 0$. It is experimentally observed that in many materials $\vec{j} = \sigma\vec{E}$ where σ is the conductivity, so that at steady state the continuity equation implies that $\vec{\nabla}\cdot\vec{E} = 0$. The laws of electrostatics remain a good approximation, so that $\nabla\cdot(\sigma\vec{\nabla}V = 0$, in regions where there are no current sources. This reduces to Laplace's equation when the conductance is uniform. With this information we can find the relation between the conductance σ and the resistance R appearing in Ohm's law $V = IR$. Typical problems are slab, concentric cylinder and concentric sphere configurations. The conductance itself may be related to the scattering time, τ of carriers in a material. Using $\vec{j} = nq\vec{v}_D$ we find that $\sigma = nq^2\tau/m$.

Faraday's law: Faraday's law states that a changing magnetic flux induces an electric field that tries to oppose the change in flux, so for one loop $\mathcal{E} = \oint \vec{E}\cdot d\vec{l} = -\frac{\partial \phi_B}{\partial t}$ or in differential form $\vec{\nabla}\wedge\vec{E} = -\frac{\partial \vec{B}}{\partial t}$. In circuits where there are N loops (and no emf from mutual inductance), we have $\mathcal{E} = -Nd\phi/dt = -Ldi/dt$. If emf from mutual inductance occurs, due to second current I, then $\mathcal{E} = -Ldi/dt - MdI/dt$. Examples include motional emf in the form of bars moving along rails and rotating coils. If a conducting rod moves through a magnetic field but is not in a conducting circuit, the charges move under the Lorentz force until the electric force on the charges balances the magnetic force. Another example is a loop that is moved away from a current carrying wire. Transformers provide a more complex example where both self-inductance and mutual inductance are important. Also know the laws for transients in charging and discharging LR and LC circuits.

The energy stored in magnetic fields may be written in many ways, with three of the most important being,

$$U = \frac{1}{2}Li^2 = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{1}{2}\int \vec{j}\cdot\vec{A}d\tau \tag{6}$$

The energy of a dipole in a uniform magnetic field is $U = -\vec{m}\cdot\vec{B}$.

Maxwell's displacement term: By considering a charging capacitor it is easy to see that Ampere's law is incorrect for time varying currents. To discover how this may be corrected, consider a time derivative of Gauss's law which yields, $dq/dt = \epsilon_0 d\phi_E/dt$. This is called Maxwell's displacement current and correctly modifies Ampere's law to the case of time varying currents. The Ampere-Maxwell equation is then, $\oint \vec{B}\cdot d\vec{l} = \mu_0 i + \mu_0\epsilon_0\frac{\partial \phi_E}{\partial t}$ which leads to the differential form, $\vec{\nabla}\wedge\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$. A standard problem is to calculate the magnetic field between the plates of a capacitor, as the capacitor charges.

Maxwell's equations and linearly polarized electromagnetic waves: The differential form of Maxwell's equations for the electric and magnetic fields in vacuum, including source terms, are (know the integral and differential forms of these equations and how to derive the differential forms from the integral forms);

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0;\quad \vec{\nabla}\cdot\vec{B} = 0;\quad \vec{\nabla}\wedge\vec{E} = -\frac{\partial \vec{B}}{\partial t};\quad \vec{\nabla}\wedge\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t};\qquad \text{(in free space)} \tag{7}$$

Electromagnetic waves occur even when there are no sources, so we set $\rho = 0$, $\vec{j} = 0$. We then isolate $\vec{E}$ and $\vec{B}$ from the last two equations. Using the vector identity, $\nabla\wedge(\nabla\wedge\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$ along with $\nabla\cdot\vec{E} = \nabla\cdot\vec{B} = 0$, we find (know how to derive this), $\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ and $\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$. These are wave equations and if we define $\vec{k}$ as the vector direction of motion of an EM wave and $\hat{n}$ to be the direction of linear polarization (this is the direction of the electric field) then the electric and magnetic fields of the EM wave satisfying these wave equations is,

$$\vec{E}(\vec{r},t) = E_0\hat{n}Cos[\vec{k}\cdot\vec{r} - \omega t + \delta];\quad \vec{B}(\vec{r},t) = \frac{1}{c}\hat{k}\wedge\vec{E}. \tag{8}$$

The vectors $\vec{k}$, $\hat{n}$ and $\vec{B}$ are perpendicular to each other. The angular frequency $\omega = 2\pi/T$ and the wave number $\vec{k}$ are related by $\omega = c|k|$, where $c^2 = 1/\mu_0\epsilon_0$ where c is the speed of light.