

**PHY831 - Subject Exam Dec. 14th 2011, 10am - 1pm**  
*Answer all questions (total points available is 100). Time for exam - 3 hours*

**Name:**

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \left(\frac{\pi}{a}\right)^{1/2} e^{\frac{b^2}{4a}}; \quad \int_0^{\infty} dx x^n e^{-ax^2} = \frac{1}{2a^{(n+1)/2}} \Gamma\left(\frac{n+1}{2}\right) \quad (1)$$

$$\int \frac{dx}{(1+x^2)^{1/2}} = \text{Sinh}^{-1}(x); \quad \int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \quad (2)$$

where  $\Gamma(s) = (s-1)!$  for  $s$  a positive integer, and  $\zeta(2) = \frac{\pi^2}{6}$ ,  $\zeta(3) = 1.202\dots$ ,  $\zeta(4) = \pi^4/90$ .

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z; \quad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z; \quad (3)$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1; \quad \left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_y + \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z; \quad (4)$$

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = T \left(\frac{\partial S}{\partial T}\right)_{V,N}; \quad C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = T \left(\frac{\partial S}{\partial T}\right)_{P,N}, \quad (5)$$

$$\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_{S,N}; \quad \kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_{T,N}; \quad \alpha_P = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_{P,N} \quad (6)$$

$$\frac{U}{L^d} = \frac{c_d}{h^d} \int_0^{\infty} dp p^{d-1} (cp^s) \frac{ze^{-\beta cp^s}}{1 - ze^{-\beta cp^s}} = \frac{d k_B T}{s \lambda^d} g_{1+d/s}(z); \quad P = \frac{s}{d} \frac{U}{L^d} \quad (7)$$

$$I(r, \nu \rightarrow \infty) = \int_0^{\infty} dx x^{r-1} \frac{ze^{-x}}{1 + ze^{-x}} = \frac{1}{r} \int_0^{\infty} dx x^r \frac{e^{x-\nu}}{(e^{x-\nu} + 1)^2} = \frac{1}{r} [\nu^r + \frac{\pi^2}{6} r(r-1) \nu^{r-2} + \dots] \quad (8)$$

$$H = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j - \sum_i h_i S_i \rightarrow \text{(for spin } \pm 1) \quad Z_{MF} = 2^N e^{-\frac{1}{2}\beta \sum_{ij} J_{ij} m_i m_j} \prod_{i=1}^N (\text{Cosh}(\beta \sum_j J_{ij} m_j + \beta h_i)). \quad (9)$$

$$Z_{vdw} = \frac{q^N}{N! \lambda^{3N}}; \quad \text{where } q = (V - Nb) e^{aN/(V k_B T)}; \quad \frac{Pv}{k_B T} = \sum_{l=1}^{\infty} a_l(T) \left(\frac{\lambda^3}{v}\right)^{l-1} \quad (\text{virial expansion}) \quad (10)$$

$$g_{GL} = \int dV \left[ \frac{1}{2m} |(-i\hbar\nabla - qA)\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2} |\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B \cdot H \right]. \quad (11)$$

$$\frac{\delta g}{\delta \psi^*} = a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m} (-i\hbar\nabla - qA)^2\psi = 0; \quad (12)$$

$$\frac{\delta g}{\delta A} = 0 \rightarrow j_s = \frac{-iq\hbar}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) - \frac{q^2}{m} A |\psi|^2 = \frac{q}{m} |\psi|^2 (\hbar \nabla S - qA) = q |\psi|^2 v_s \quad (13)$$

$$H_{MF} - \mu N = \sum_{\vec{k}\sigma} (\epsilon_{\vec{k}\sigma} - \mu) a_{\vec{k}\sigma}^\dagger a_{\vec{k}\sigma} - \sum_{\vec{k}} (\Delta_{\vec{k}} a_{\vec{k}\uparrow}^\dagger a_{-\vec{k},\downarrow}^\dagger + \Delta_{\vec{k}}^* a_{-\vec{k},\downarrow} a_{\vec{k}\uparrow} - b_{\vec{k}}^* \Delta_{\vec{k}}); \quad (14)$$

$$H_{MF} - \mu N = \sum_{\vec{k}} (\epsilon_{\vec{k}} - \mu - E_{\vec{k}} + \Delta_{\vec{k}} b_{\vec{k}}^*) + \sum_{\vec{k}} E_{\vec{k}} (\gamma_{\vec{k}\uparrow}^\dagger \gamma_{\vec{k}\uparrow} + \gamma_{\vec{k}\downarrow}^\dagger \gamma_{\vec{k}\downarrow}), \quad (15)$$

$$\Delta_{\vec{k}} = -\sum_{\vec{l}} V_{\vec{k}\vec{l}} b_{\vec{l}}; \quad b_{\vec{k}} = \langle a_{-\vec{k},\downarrow} a_{\vec{k}\uparrow} \rangle = \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}} (1 - 2f(E_{\vec{k}})); \quad E_{\vec{k}} = ((\epsilon_{\vec{k}} - \mu)^2 + |\Delta_{\vec{k}}|^2)^{1/2} \quad (16)$$

**Problem 1. (16 points)**

(i) (10 points) Show that the fluctuations of the energy  $E$  in the canonical ensemble are given by,

$$\delta E^2 = \langle E^2 \rangle - \langle E \rangle^2 = k_B T^2 C_V. \quad (17)$$

where  $\langle O \rangle$  indicates the ensemble average of the operator  $O$ ,  $C_V$  is the heat capacity at constant volume,  $k_B$  is Boltzmann's constant and  $T$  is temperature.

(ii) (6 points) Using this result explain why  $\delta E / \langle E \rangle$  is proportional  $1/N^{1/2}$ , where  $N$  is the number of particles or spins in the system. What is the significance of this result for the connections between statistical physics and thermodynamics?

**Solutions**

(i) The internal energy and its second moment are found from the canonical partition function using,

$$\langle E \rangle = U = \frac{-1}{Z} \frac{\partial Z}{\partial \beta}; \quad \text{and} \quad \langle E^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} \quad (18)$$

where  $Z = \sum_E e^{-\beta E}$ . From these expressions we find the result,

$$(\delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2 = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \left( \frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)^2 = \frac{\partial^2 \ln(Z)}{\partial \beta^2}. \quad (19)$$

where  $\langle E^n \rangle = \sum_E E^n e^{-\beta E}$ . Therefore,

$$(\delta E)^2 = - \left( \frac{\partial U}{\partial \beta} \right)_{V,N} = - \left( \frac{\partial U}{\partial T} \right)_{V,N} \left( \frac{\partial T}{\partial \beta} \right)_{V,N} = k_B T^2 C_V \quad (20)$$

(ii) Since the energy and the specific heat are extensive, we have,

$$\frac{\delta E}{E} \propto \frac{1}{N^{1/2}} \quad (21)$$

as required by the central limit theorem. This means that the relative deviations in the internal energy calculated in the canonical ensemble have fluctuations of order  $N^{1/2}$ . The probability distribution of the internal energy in a system with  $N$  particles is then a sharply peaked function with its mean equal close to the thermodynamic value of the internal energy  $U = \langle E \rangle$ . Thermodynamic properties can then be calculated by taking appropriate averages over statistical ensembles.

**Problem 2. (16 points)**

An ideal monatomic Bose gas of particles of mass  $m$  has dispersion relation  $\epsilon_p = cp^s$ , where  $c$  is a constant (e.g. for a non-relativistic gas  $c = \hbar^2/2m$ ,  $s = 2$ ), so that its thermodynamics in  $d$  dimensions is described by,

$$\frac{U}{L^d} = \frac{d}{s} \frac{k_B T}{\lambda^d} g_{1+d/s}(z); \quad \frac{N}{L^d} = \frac{1}{\lambda^d} g_{d/s}(z) + \frac{1}{L^d} \frac{z}{1-z}; \quad P = \frac{s}{d} \frac{U}{L^d}; \quad g_y(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^y} \quad (22)$$

where  $N$  is the number of particles and  $L^d$  is the size of the hypercubic container in  $d$  dimensions. The fugacity  $z = e^{\beta\mu}$ , where  $\mu$  is the chemical potential.

**(i) (5 points)** Find the critical dimension  $d_c$  as a function of  $s$  above which there is a Bose condensation phase transition at finite temperature.

**(ii) (6 points)** For cases where there is a finite temperature Bose condensation transition, find an expression for the order parameter (the Bose condensate fraction  $N_0/N$ ) as a function of temperature, where  $N_0$  is the number of particles in the ground state.

**(iii) (5 points)** Find the critical exponent describing the scaling behavior of the order parameter near the critical point. Does the order parameter critical exponent depend on the spatial dimension? Is this consistent with the universality hypothesis?

**Solutions**

(i) Bose condensation occurs when a finite fraction of the particles are in the ground state,  $N_0$ . We write

$$N = \frac{L^d}{\lambda^d} g_{d/s}(z) + \frac{z}{1-z} = N_0 + N_1 \quad (23)$$

where  $N_1$  is the number of particles in the excited states of the Bose gas. Mathematically condensation is identified by the condition  $\mu \rightarrow 0$ ,  $N_1 = N$ , which leads to the critical relation,

$$N_1 = N = \frac{L^d}{\lambda_c^d} g_{d/s}(1) \quad (24)$$

The requirement for a finite temperature phase transition is therefore that  $g_{d/s}(1)$  be finite. If  $g_{d/s}(1)$  is not finite, there is no finite temperature Bose condensation. From the definition of  $g_y$  in Eq. (22), it is seen that provided  $d/s > 1$ ,  $g_y(1)$  remains finite, so there can be a finite temperature condensation transition, provided the critical condition  $d_c > s$  is satisfied.

(ii) The critical temperature of the transition is found from Eq. (23)

$$\lambda_c = \frac{L[g_{d/s}(1)]^{1/d}}{N^{1/d}}; \quad \text{where} \quad \lambda_c^2 = \frac{h^2}{2\pi m k_B T_c} \quad (25)$$

The order parameter for the transition is  $N_0/N$  so that for  $T < T_c$ ,

$$\frac{N_0}{N} = 1 - \frac{L^d}{N\lambda^d} g_{d/s}(1) = 1 - \left(\frac{\lambda_c}{\lambda}\right)^d = 1 - \left(\frac{T}{T_c}\right)^{d/2} \quad (26)$$

(iii) To find the critical behavior, we expand near the critical point using  $T/T_c = 1 - (T_c - T)/T_c = 1 - t$ , so that,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{d/2} = 1 - (1 - t)^{d/2} \approx \frac{d}{2} t \quad (27)$$

The order parameter critical exponent for the ideal Bose gas condensation is thus  $\beta = 1$ .

**Problem 3. (16 points)**

(i) (5 points) On a  $P-v$  diagram, where  $v = V/N$ , plot the isotherms of the van der Waals equation. Also sketch the spinodal lines which are defined by  $\partial P/\partial v = 0$ . Here  $P$  is pressure,  $V$  is volume and  $N$  is the number of particles in the system. Sketch the boundaries of the co-existence region and state the Maxwell condition used to construct them.

(ii) (8 points) Using the fact that the liquid gas transition is in the same universality class as the Ising model, find the upper and low critical dimensions for the liquid-gas phase transition.

(iii) (3 points) What is the order parameter for the liquid-gas phase transition - give an expression for the expected scaling behavior of the order parameter near the critical point of the liquid-gas system. What is the mean field value of the order parameter exponent?

**Solutions**

(i) The Maxwell condition is given by,

$$\int_{v_l}^{v_g} p dv = p^*(v_g - v_l)$$

(ii) The lower critical dimension is determined by the energy of domain walls between the phases in the material. In the case of the Ising model, the domain wall energy is,

$$E_{dw} \approx 2\sigma L^{d-1} \quad (28)$$

where  $\sigma$  is the surface energy of the domain wall. For any dimension  $d > 1$ , the domain wall energy increases with the size of the domain, so we expect there to be a finite temperature phase transition. The lower critical dimension of the Ising model and the liquid-gas transition is then  $d_{lc} = 1$ .

The upper critical dimension is deduced using the Lifshitz criterion which considers the ratio  $C(\xi)/m^2$ , where  $C(r)$  is the pair correlation function and  $m$  is the order parameter. If this ratio diverges, then fluctuations are important and corrections to mean field theory are required. We then consider,

$$\frac{C(\xi)}{m^2} \approx \frac{\xi^{-(d-2+\eta)}}{t^{2\beta}} \approx t^{(d-2+\eta)\nu-2\beta} \quad (29)$$

If the exponent  $(d-2+\eta)\nu-2\beta$  is negative, the fluctuations dominate and mean field theory is not correct, while if the exponent is positive fluctuations are not important and mean field theory is OK. The critical dimension is given by,

$$(d_{uc} - 2 + \eta)\nu - 2\beta = 0 \quad (30)$$

Using the mean field exponents  $\eta = 0, \nu = 1/2, \beta = 1/2$  that are correct for the liquid-gas transition, we find  $d_{uc} = 4$ .

(iii) The order parameter for the liquid gas phase transition is the difference in specific volumes of the gas and liquid phases on the co-existence curve  $v_g - v_l$ . Where  $v_l$  and  $v_g$  are the values on the co-existence curve. The order parameter behavior near the critical point scales as,

$$v_g - v_l \sim |T - T_c|^\beta \quad (31)$$

where  $\beta$  is the order parameter critical exponent and takes the value  $\beta = 1/2$  within van der Waals theory, which is a mean field theory for the liquid gas transition.

**Problem 4. (16 points)**

Consider  $N$  electrons moving in a two dimensional plane of area  $A = L^2$ . A magnetic field,  $B$ , is applied along the normal to the plane. The Bohr magneton is defined as  $\mu_B = e\hbar/(2m_e)$ . The magnetic moment of the electron is, to a good approximation, equal to  $\mu_B$ .

(i) **(3 points)** Find the lowest magnetic field  $B_1$  that is strong enough so that all of the electrons can be accommodated in the lowest Landau level (Hint: the degeneracy of a spinless Landau level is  $g = BAe/h$ , where  $h$  is Planck's constant).

(ii) **(3 points)** Find the field  $B_2$  above which all of the electrons are in the up spin sub-level of the first Landau level.

(iii) **(10 points)** Taking into account both the paramagnetic and diamagnetic contributions, find an expression for the magnetization of the free electron gas at temperature  $T = 0K$  as a function of the magnetic field for  $B > B_1$ . Sketch the behavior of the magnetization as a function of the field.

**Solution.**

(i) The lowest Landau level is split by the application of a magnetic field into two sublevels with energies,

$$E_- = \frac{1}{2}\hbar\omega_c - \mu_s B; \quad E_+ = \frac{1}{2}\hbar\omega_c + \mu_s B \quad (32)$$

where  $\omega_c = eB/m$ . The degeneracy of the Landau level is  $BAe/h$ . If there are  $N$  electrons, the first field ( $B_1$ ) at which this number of electrons can be accommodated in the lowest Landau level is found from the relation  $2g = N$ , so we have,

$$2g = 2\frac{B_1 Ae}{h} = N; \quad \text{or} \quad B_1 = \frac{h N}{2e A} \quad (33)$$

where  $A$  is the area of the sample.

(ii) The field at which all of the electrons are in the up spin sub-band is given by,

$$g = \frac{B_2 Ae}{h} = N; \quad \text{or} \quad B_2 = \frac{h N}{e A} \quad (34)$$

(ii) Since we are in the ground state we can consider just the energy and find the magnetization using

$$M = -\frac{\partial E_G}{\partial B} \quad (35)$$

This is true as

$$M = \frac{1}{\beta} \frac{\partial \ln(Z)}{\partial B} = -\frac{\partial F}{\partial B} \rightarrow -\frac{\partial E_G}{\partial B} \quad (36)$$

We can consider the two contributions to the magnetization separately. The diamagnetic contribution comes from the diamagnetic part of the energy  $N\hbar\omega_c/2$ , so

$$M_D = -\frac{\partial}{\partial B}(N\frac{\hbar e B}{2m}) = -N\frac{e\hbar}{2m} = -N\mu_B \quad \text{where} \quad \mu_B = \frac{e\hbar}{2m_e} \quad (37)$$

where  $\mu_B$  is the Bohr magneton and the spin magnetic moment of the electron is approximately  $\mu_B$ .

The paramagnetic contribution to the magnetization comes from the filling of the up and down levels as a function of applied field  $B$ . The paramagnetic (spin) contribution to the magnetization is  $M_P = \mu_B(N_\uparrow - N_\downarrow)$ . At  $B_1$ ,  $N_\uparrow = N_\downarrow$ , so  $M = 0$ . If the magnetic field is increased further, the degeneracy of the ground state continues to increase linearly with the field. For fields greater than this field all of the spins are in the up spin state, so  $M_P = N\mu_B$ . We then find,

$$M = M_D + M_P = -N\mu_B + N\mu_B = 0 \quad \text{for} \quad B > B_2 \quad (38)$$

In the field range  $B_1 < B < B_2$ , the number of spins in the up spin sublevel is  $g$ , while the number in the down spin sublevel is  $N - g$ , so the paramagnetic contribution is,

$$M_P = (g - (N - g))\mu_B = (2g - N)\mu_B = (2\frac{BAe}{h} - N)\mu_B \quad (39)$$

so the magnetization in this regime is,

$$M = M_D + M_P = (2\frac{BAe}{h} - N)\mu_B - N\mu_B = -2N\mu_B(1 - \frac{B}{B_2}) \quad B_1 < B < B_2 \quad (40)$$

The magnetic response is diamagnetic for  $B_1 < B < B_2$ , while there is no magnetic moment for  $B > B_2$ , within this approximation.

**Problem 5. (18 points)**

(i) (8 points) (a) Sketch the field-temperature phase diagram for a type II superconductor indicating the Meissner, mixed and normal phases; (b) Sketch the behavior of the superconducting gap,  $\Delta$ , as a function of temperature, at zero applied magnetic field, for an s-wave BCS superconductor. (c) Write down a mathematical expression for the scaling behavior of the gap near the critical temperature,  $T_c$ ; (d) For an s-wave BCS superconductor, sketch the density of states of the quasiparticle excitations from the BCS ground state as a function of energy, near the Fermi energy,  $\epsilon_F$ .

(ii) (4 points) Within London theory, which is valid in the extreme type II limit, the Helmholtz free energy per unit length of an isolated flux quantum in a type II superconductor is approximately,

$$f_1 \approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \text{Ln}\left(\frac{\lambda}{\xi}\right) \quad (41)$$

Using this result find an expression for the lower critical field  $H_{c1}$  of a type II superconductor. Here  $\phi_0$  is the flux quantum,  $\lambda$  is the penetration depth,  $\xi$  is the coherence length,  $\mu_0$  is the permeability.

(iii) (6 points) By linearizing the G-L equation, find an expression for the upper critical field of the superconductor,  $H_{c2}$ , in terms of the parameters in the G-L theory. To write the result in its usual form you can use  $\xi^2 = \hbar^2/(2m|a|)$ ,  $\phi_0 = h/q = h/2e$ . (hint: the eigenvalues of a Cooper pair in a magnetic field are,  $\epsilon_{n,k_z} = (n + 1/2)\hbar\omega_c + \hbar^2k_z^2/2m$ )

**Solutions**

(i)

(c) Near the superconducting transition temperature, the gap behaves as,

$$\Delta \approx (T_c - T)^{1/2} \quad (42)$$

(ii) The Gibb's free energy per unit volume for an isolated flux quantum in a homogeneous superconductor is,

$$G_1 = F_1 - B \cdot H \quad (43)$$

Integrating over the area, this gives,

$$g_1 = f_1 - \phi_0 H \quad (44)$$

Flux penetration is favorable when  $g_1 < 0$ , which leads to the condition,

$$H_{c1} = \frac{f_1}{\phi_0} = \frac{\phi_0}{4\pi\mu_0\lambda^2} \text{Ln}\left(\frac{\lambda}{\xi}\right) \quad (45)$$

(iii) The upper critical field is found by solving the G-L equation in the linear limit

$$\frac{1}{2m}(-i\hbar\nabla - qA)^2\psi = |a(T)|\psi \quad (46)$$

where we interpret  $|a(T)|$  as a positive eigenvalue. We want to find the largest field for each value of  $|a(T)|$ , which corresponds to a distance from the critical point as  $a(T) \sim -|T_c - T|$ .

The eigenvalues are given by,

$$\epsilon_{n,k_z} = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2k_z^2}{2m} \quad (47)$$

where  $\omega_c$  is the cyclotron frequency that may be found using,

$$m\omega^2 r = qvB = q\omega rB; \quad \text{so that} \quad \omega_c = \frac{qB}{m} \quad (48)$$

The upper critical field is the largest  $B$  for which there is a solution to this equation. The largest  $B$  solutions occurs for  $n = k_z = 0$ , so that,

$$\frac{\hbar\omega_c}{2} = |a(T)|; \quad \text{and hence} \quad H_{c2} = \frac{2m|a|}{\hbar q\mu_0} = \frac{\phi_0}{2\pi\mu_0\xi^2} \quad (49)$$

**Problem 6. (18 points)**

A spin half Ising model with three spin interactions on a triangular lattice has Hamiltonian,

$$H = - \sum_{ijk\epsilon\Delta} JS_iS_jS_k \quad (50)$$

where the sum is over all nearest neighbor triangles on an infinite triangular lattice, and the interaction is ferromagnetic  $J > 0$ .

(i) (6 points) Using a leading order expansion in the fluctuations (i.e. write  $S_i = m_i + (S_i - m_i)$  and expand to leading order in the fluctuations), find the mean field Hamiltonian for this problem (Here  $m_i = \langle S_i \rangle$  is the magnetization at site  $i$ ).

(ii) (6 points) Using the mean field Hamiltonian and assuming a homogeneous state where  $m_i = m$ , find an expression for the mean field Helmholtz free energy, and the mean field equation for this problem.

(iii) (6 points) Taking  $J = 1$ , sketch the behavior of the solutions to the mean field equation as a function of temperature. Does the non-trivial solution move continuously toward the  $m = 0$  solution as the temperature increases? Is the behavior of the order parameter at the critical temperature discontinuous or continuous? Do you expect the correlation length to diverge at the critical point in this problem?

**Solutions**

(i) We start by writing a three spin term in terms of an expansion in the fluctuations,

$$S_iS_jS_k = [m_i + (S_i - m_i)][m_j + (S_j - m_j)][m_k + (S_k - m_k)] \quad (51)$$

Expanding to leading order in the fluctuations, we then have,

$$S_iS_jS_k \approx m_im_jm_k + (S_i - m_i)m_jm_k + m_i(S_j - m_j)m_k + m_im_j(S_k - m_k) \quad (52)$$

Since the interaction is ferromagnetic we can make the uniform assumption  $m_i = m$ , which yields,

$$H_{MF} = - \sum_{ijk\epsilon\Delta} [-2Jm^3 + m^2(S_i + S_j + S_k)] = 2JNm^3 - 3Jm^2 \sum_i S_i \quad (53)$$

(ii) The mean field partition function is then,

$$Z_{MF} = e^{-2\beta JNm^3} [2\text{Cosh}(3\beta Jm^2)]^N \quad (54)$$

which leads to the free energy,

$$F_{MF} = -k_B T \ln Z_{MF} = 2JNm^3 - k_B TN \ln [2\text{Cosh}(3\beta Jm^2)] \quad (55)$$

and the mean field equation is,

$$m = \text{Tanh}(3\beta Jm^2) \quad (56)$$

The mean field equation can be found in several ways, for example by doing the variation  $\delta F/\delta m = 0$ , or by finding the expectation  $\langle S_i \rangle$  using the mean field Hamiltonian.

(iii) The first thing to note is that  $\text{Tanh}(3\beta Jm^2)$  for ferromagnetic interactions (i.e.  $J > 0$ ) is always positive, so there are no negative  $m$  solutions to the mean field equation. This is completely different than the pair interaction case where there is symmetry in the solutions for positive and negative  $m$ .

By plotting the graphs of  $m$  and  $\text{Tanh}(3\beta Jm^2)$ , it is evident that at large values of  $K = \beta J$  (i.e. low temperature), there are two positive  $m$  solutions as well as the solution at  $m = 0$ . However as  $K$  decreases (i.e.  $T$  increases) there is a critical point at which the two non-trivial positive  $m$  solutions merge. This defines the critical point. At this point  $m$  is finite. For  $T > T_c$  the positive  $m$  solution disappears so there is a discontinuous jump in the magnetization from a finite value to zero at  $T_c$ . The transition in this model is then first order, in contrast to the pair interaction case where the transition is continuous. The correlation length at the transition found here is then expected to remain finite at the critical point.