Statistical Physics (PHY831): Part 4: Ginzburg-Landau theory, modeling of dynamics and scaling in complex systems

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Part 4: (9 lectures), (H, PB) Ginzburg-Landau theory of superconductivity. Scaling theory of second order phase transitions. Lower critical dimension, upper critical dimension. Renormalization group ideas and methods. Equilibrium and non-equilibrium dynamics. Diffusion, Langevin theory. Conserved and non-conserved dynamics. Fractals and percolation.

Midterm 4, Lecture 42 (Friday Dec. 9)

I. GINZBURG LANDAU THEORY OF SUPERCONDUCTORS

A. Introduction

London theory was developed by Fritz London in 1935 to describe the Meissner effect. This theory leads to the introduction of the penetration depth to describe the extent of magnetic field penetration, λ into type I superconductors. The penetration depth is also important in type II superconductors and describes the extent of flux penetration near vortices as well as at surfaces.

Prior to his studies of superconductivity, Landau had developed a simple mean field theory to describe phase transitions. Ginzburg added a term to describe fluctuations which also enables description of inhomogenious systems. Ginzburg-Landau (GL) theory is a field theory and provides a systematic phenomenological approach to many body systems. The GL theory introduces a second length, the healing length or coherence length ξ. Below we first introduce the G-L theory and we show how the length ξ emerges.

Before proceeding it is important to note that the analysis of London theory and LG theory below uses q for charge, m for mass and n_s for the number density of superconducting electrons. In all superconductors found so far $q = 2e$ is the charge of the fundamental Bosons (Cooper pairs), $n_c = n_s/2$ is the number density of cooper pairs and m is the effective mass of Cooper pairs. In some materials m can be significantly different than $2m_e$ due to band structure effects.

Ginzburg-Landau theory, which was published in 1950, does a good job of describing the electromagnetic properties of superconductors, including vortex effects and the effect of pinning on these vortices. One of the key successes of the Ginzburg-Landau theory is its prediction of the distinction between type I and type II superconductors that have very different electromagnetic properties. Flux penetrates type II superconductors in the form of quantized vortices with flux $\phi_0 = h/2e$. The reason for flux quantization is purely quantum mechanical, as it arises through the requirement that the wavefunction of the superconductor be single valued at every point in space.

B. Ginzburg-Landau theory: Zero field, the healing length ξ

Landau theory is a phenomenological mean field theory to describe behavior near a phase transition. In the case of a superconductor, where the superconducting electrons are described by a "macroscopic" wavefunction, $\psi(\vec{r})$, the Landau free energy is,

$$f_L = f_s(T) - f_n(T) = a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 \tag{1}$$

For $T < T_c$, $a(T) < 0$ and $b(T) > 0$. If we assume that the superconductor is uniform, minimizing f_L with respect to ψ yields the physical solution,

$$|\psi_\infty|^2 = \frac{-a}{b} \ ; \quad f_L(|\psi_\infty|) = \frac{-a^2}{2b}, \tag{2}$$

The free energy of a weak coupling isotropic BCS systems can be reduced to,

$$f_{BCS} = f_s - f_n = -N(\epsilon_F)\Delta(T)^2[\frac{1}{2} + ln(\frac{\Delta(0)}{\Delta(T)})] + \frac{\pi^2}{3}N(\epsilon_F)(k_BT)^2 - 4N(\epsilon_F)k_BT\int_0^{\hbar\omega_c} ln(1+e^{-\beta E})d\epsilon \tag{3}$$

which reduces to $N(E_f)\Delta^2(0)/2$ at zero temperature, so that we can make the connection to the Landau theory through $a^2(0)/2b = N(E_f)\Delta^2(0)/2$. Expanding for small Δ, it can be shown that near the critical point, $f_{BCS} \propto |T - T_c|^2$. We then find that near the critical temperature $a^2(t)/2b = |T - T_c|^2$, so that $a(T) \propto (T - T_c)$ as proposed by Landau. This is also consistent with mean field theory where the specific heat exponent is $\alpha = 0$

In order to add fluctuations (local variations in the wavefunction) to this model, Ginzburg suggested adding a term proportional to $|\nabla\psi(\vec{r})|^2$. There are many ways to motivate this term. Firstly it is the kinetic energy term in quantum mechanics. Secondly it is the lowest order fluctuation term allowed by the symmetry of the order parameter. Thirdly a term like this can be derived directly from the nearest neighbour exchange model of magnetism. The prefactor of this term is often called the "stiffness" as it controls the ability of the material to fluctuate. Adding this term to the free energy (1), we have the Ginzburg-Landau theory in zero field,

$$f_{GL} = f_s(T) - f_n(T) = \int [\frac{\hbar^2}{2m}|\nabla\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4]dV \tag{4}$$

Notice that this has the same form as the Gross-Pitaevskii equation for the interacting Bose gas, as studied for example in atom traps. Only the interpretation of the parameters is different. Minimizing f_{GL} with respect to $\psi^*(\vec{r})$, yields,

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r}) + a(T)\psi(\vec{r}) + b(T)\psi(\vec{r})|\psi(\vec{r})|^2 = 0 \tag{5}$$

To illustrate the origin of the Ginzburg-Landau coherence length, define $s(x) = \psi(\vec{r})/\psi_\infty$ and consider equation (5) in one dimension, then we have (using $|\psi_\infty|^2 = -a/b = |a|/b$), and considering only the amplitude we have,

$$\xi^2\frac{d^2s(x)}{dx^2} + s(x) - s(x)^3 = 0 \quad \text{where} \quad \xi = (\frac{\hbar^2}{2m|a(T)|})^{1/2} = \frac{\hbar v_F}{\pi\Delta} \tag{6}$$

The last expression is the result of a BCS calculation using a similar approach (v_F is the Fermi velocity). It is clear from this equation that ξ is a length over which the superconducting order parameter fluctuates. It is proportional to the correlation length so we find the correlation length exponent is $\nu = 1/2$, which is the mean field value.

With this theory we can also study the typical length over which superconducting order decays near an air or insulating interface. An interesting solution which shows this explicitly is an interface consisting of an insulator on one side and the other side a superconductor. We consider the interface to be planar, at the origin, and its normal to be in the $\hat{x}$ direction. The boundary conditions that we need are that $s(x \to \infty) = 1$, and $s(x \to -\infty) = 0$. This equation has a rather complicated exact solution, however, the behavior of interest can be found by considering a solution $s = 1 - g$ where g is small. A first order expansion in g of Eq. (6) gives,

$$-\xi^2\frac{d^2g}{d_x^2} = -2g(x); \qquad \textit{so that} \qquad g(x) \approx e^{\pm\sqrt{2}x/\xi} \tag{7}$$

showing that the order parameter varies on length scales of order ξ as expected.

C. Adding a magnetic field to GL theory

Many applications of GL theory are to the analysis of the effects of an applied magnetic field. In the presence of a magnetic field, the Helmholtz free energy above must be extended in two ways. Firstly, the momentum operator $p = -i\hbar\nabla$ is replaced by $p \to -i\hbar\nabla - qA$, where A is the vector potential associated with the magnetic field B, and $-q$ is the charge of the cooper pair. We use the following definitions $m = 2m^*$ (where m^* is the effective mass of the electron), $q = 2e$, $|\psi|^2 = n_s$, where n_s is the density of superconducting electrons. Secondly, the field energy has to be added to the free energy. These modifications lead to the expression,

$$f_{GL} = \frac{1}{2m}|\int dV[(-i\hbar\nabla - qA)\psi(\vec{r}))|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} - \frac{\mu_0 H^2}{2}] \tag{8}$$

The term $B^2/2\mu_0$ is the magnetic field energy inside the superconductor while $\mu_0 H^2/2$ is the field energy in the normal state.

However it is wrong to use the Helmholtz energy in calculations where the external field controls the electrodynamics. This is because we must take into account the amount of energy required to set up the applied field as well. The free energy we need to use is the Gibb's free energy $g = f - \mu_0 H \cdot M$. Where M is the magnetisation. If we take the normal state magnetisation to be zero, and use, $B = \mu_0(H + M)$, we find that the Gibb's free energy is,

$$g_{GL} = \frac{1}{2m}[\int dV|(-i\hbar\nabla - qA)\psi(\vec{r}))|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B \cdot H]. \tag{9}$$

Note that in many superconducting texts and papers M is taken to have the units of B. I am using the conventional magnetostatics definition. Within mean-field theory, we assume that the order parameter takes on a value which optimizes the above free energy. In this expression we can optimize the wavefunction and the field. We thus do a variation with respect to the wavefunction (or ψ^*) to produce the GL equation. In addition we do a variation with respect to the vector potential A which, as we shall see, leads to an expression for the diamagnetic current. Using the Euler-Lagrange equations to do the variation with respect to $\psi*$ yields,

$$a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - qA)^2\psi = 0 \tag{10}$$

A variation with respect to the vector potential yields (and using $\mu_0 j = \nabla \wedge B$, and the gauge $\nabla \cdot A = 0$), we find,

$$j_s = \frac{-iq\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m}A|\psi|^2$$

$$= \frac{q}{m}|\psi|^2(\hbar\nabla S - qA) = q|\psi|^2 v_s \tag{11}$$

The electromagnetic properties of superconductors are determined by Eqs. (9-11)) in combination with Maxwell's equations. A good starting point to understand the electrodynamics of superconductors is London theory, that sets $|\psi| = constant = |\psi_\infty|$, that is, the amplitude of the order parameter is assumed to be a constant.

The London limit

London argued that supercurrent travels ballistically and that its kinetic energy should be included in the free energy of the superconducting state. He took the free energy of the superconducting state to be,

$$f_{London} = \int (\frac{1}{2}n_c m v_s^2 + \frac{1}{2\mu_0}B^2)dV. \tag{12}$$

The superconducting current density is given by,

$$j_s = n_c q v_s = n_s e v_s. \tag{13}$$

Using Maxwell's equation

$$\nabla \wedge B = \mu_0 j + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{14}$$

with $\partial E/\partial t = 0$ as we are first considering steady state solutions, yields

$$f_{London} = \frac{1}{2\mu_0}\int dV[\lambda^2(\nabla \wedge B)^2 + B^2] \tag{15}$$

where

$$\lambda = (\frac{m}{n_c q^2 \mu_0})^{1/2}(MKS) \quad \text{or} \quad \lambda = (\frac{mc^2}{4\pi n_c q^2})^{1/2}(CGS) \tag{16}$$

Doing a variation of this expression, ie. $\frac{\delta f_{London}}{\delta B} = 0$, and some algebra yields

$$\nabla^2 B = \frac{B}{\lambda^2} \tag{17}$$

London theory is used a great deal in studying flux states in type II superconductors.

Another view of London theory is as a special case of GL theory, where the superconducting electrons are related to the number density through $|\psi(r)|^2 = n_s(r)$. The superconducting current may be written as,

$$j_s = \frac{q\hbar}{m}|\psi|^2\nabla\phi - \frac{q^2}{m}|\psi|^2 A = n_s e v_s = n_c q v_s \tag{18}$$

This is an important equation and later we use it to demonstrate flux quantization. Here we note that in a simply connected system the gauge transformation,

$$A = A_1 + \nabla\chi, \quad \text{with} \quad \phi = \phi_1 + q\chi/\hbar \tag{19}$$

leaves B and j_s unaltered. If we take the gauge $\chi = \hbar S/q$, then we are left with,

$$j_s = \frac{-q^2}{m}|\psi|^2 A \tag{20}$$

which is often called London's equation. Assuming that the density of superconducting electrons is constant, taking the curl of this equation, using $B = \nabla \wedge A$, along with Maxwell's equation ie,

$$\nabla \wedge j_s = \frac{-q^2 n_s}{m} B \quad \text{with} \quad j_s = \frac{1}{\mu_0}\nabla \wedge B \tag{21}$$

yields,

$$B + \lambda^2(\nabla \wedge \nabla \wedge B) = 0 \tag{22}$$

where λ is given by Eq. (16). Using the identity $\nabla \wedge \nabla \wedge B) = \nabla(\nabla \cdot B) - \nabla^2 B$ and $\nabla \cdot B = 0$, we find

$$B - \lambda^2(\nabla^2 B) = 0 \tag{23}$$

which is the same as Eq. (17) above.

The first major success of London's theory is an explanation of the Meissner effect, which is most easily demonstrated in a planar geometry. Consider an applied field $B_0\hat{z}$ in vaccuum in the region $x < 0$. Consider that there is a superconductor in the region $x > 0$. We want to find the behavior of the magnetic field inside the superconductor and we do that by using London's equation. In the simple planar geometry, the solutions to London's equation (Helmholtz equation) are $exp(\pm x/\lambda)$. Clearly the physical solution is $exp(-x/\lambda)$, so that,

$$B(x) = B_0 e^{-x/\lambda} \quad \text{in the z} - \text{direction} \tag{24}$$

Note that the reduction in magnetic field is achieved by generating demagnetizing supercurrents which circulate in the xy plane to cancel out the applied field in the interior. The demagnetizing currents are found by calculating,

$$j_s = \frac{1}{\mu_0}\nabla \wedge B \tag{25}$$

London's solution ignores the variations in the magnitude of the wavefunction near the surface, so it is valid when the coherence length ξ is small.

D. The two lengths ξ and λ

The coherence length ξ describes the length scale of variations in the magnitude of ψ or the density of superconducting electrons, while λ describes the penetration depth of magnetic fields into a superconductor. We have carried out calculations that considered either an exponential variation in the magnitude of ψ over length scales of the coherence length, or exponential variations in the magnetic field and screening currents over length scale λ. Though these calculations were carried out for special cases, the two lengths we extracted are always important. In general both order parameter magnitude variations and magnetic field penetration have to be considered. The ratio of these lengths is defined to be $\kappa = \lambda/\xi$. The value of κ determines whether a superconductor is Type I ($\kappa < 1\sqrt{2}$) or type II ($\kappa > 1\sqrt{2}$).

E. The thermodynamic critical field, H_{cb}

The thermodynamic or bulk critical field H_{cb} is the field at thich the field energy is the same as the condensation free energy, f_{cond}. We calculated the condensation free energy within Landau theory (see Eq. (2)), and the BCS expression for this energy is given in Eq. (3). From these expressions, we find that $f_{cond} \propto |T - T_c|^2$ near T_c for both BCS theory and Landau theory, as expected for mean field calculations where $\alpha = 0$. The field energy is $\mu_0 H^2/2$, which is the field energy required to expel flux from the interior of a superconductor, as occurs in the Meissner phase. This calculation assumes a field applied parallel to a slab of superconductor with thickness $t >> \lambda$. In that case the thermodynamic critical field is given by,

$$\frac{\mu_0 H_{cb}^2}{2} = f_{cond} \sim |T - T_c|^2. \tag{26}$$

In this analysis we have considered only two states, the normal state with uniform flux and the Meissner phase of a superconductor. It turns out that for type I superconductors this is correct and there is a first order transition from the Meissner phase to the normal phase. The phase diagram then consists of just those two phases. If the applied field/sample geometry is not a parallel field applied to a slab, demagnetization effects can lead to complex flux penetration patterns. This regime that is sample geometry dependent is called the intermediate phase.

Type II superconductors do not make a direct transition from the Meissner phase to the normal phase, and instead make the transition through an intermediate phase called the mixed phase. To estimate the value of κ that separates Type I superconductors from type II superconductors, we consider a lamellar mixed phase, as originally considered by Landau. We consider whether it is energetically favorable to add interfaces into the material at H_{cb}. The Gibb's free energy to add an interface is approximately,

$$G_{interface}(H_{cb}) = f_{cond} L^2 \xi - \frac{1}{2}\mu_0 H^2 L^2 \lambda; \quad \text{so that,} \quad \frac{G_{interface}(H_{cb})}{L^2} = -\frac{1}{2}\mu_0 H_{cb}^2 (\lambda - \xi) \tag{27}$$

Clearly the superconducting phase at H_{cb} is unstable to the formation of interfaces provided $\lambda > \xi$, or $\kappa > 1$. Abrikosov showed that the exact critical value, $\kappa_c = 1/2^{1/2}$, that is extracted by the analysis of vortex lattices which is the correct morphology of the mixed phase.

In type I superconductors where $\kappa < 1/2^{1/2}$ there is one critical field H_{cb} at which the flux suddenly penetrates the sample (ignoring demagnetisation effects) while in type II superconductors there are two critical fields: H_{c1} when flux quanta first penetrate a sample and H_{c2} when the applied field finally destroys superconductivity (at H_{c2} the fluxons pack so densely that their cores overlap). We have already determined H_{cb} and below we shall find the two critical fields H_{c1}, H_{c2} using GL theory in the limit of large κ (extreme type II). Before doing that we need to develop a more complete understanding of an isolated vortex.

The discussion below ignores the effect of random pinning on the flux states, which is only valid in the cleanest materials. The fields we find are the "reversible" critical fields. In most type II materials pinning is important and there is considerable hysteresis in the magnetisation. In these cases, one can define a variety of irreversibility lines. All commercial magnets and proposed transmission line applications of superconductors require good flux pinning. Vortices are also called flux lines, and they experience a Lorentz force when a DC current flows through a superconductor. If there is no flux pinning, the vortices move leading to an induced emf, through Faraday's/Lenz law.

F. Flux quantization

If it is favorable to form interfaces in a superconductor, the superconductor will make as many interfaces as possible. Landau thought that these interfaces would be generated in the form of lamella. Abrikosov, in 1957, introduced the idea that flux penetrates type II superconductors in the form of quantized vortices. Vortices are quantized in type II superconductors for the same reason that circulation is quantized in superfluids. Nevertheless it is worth stating the arguments again.

(i) The wavefunction must be single valued at any location in a superconductor.

(ii) Statement (i) implies that along any closed loop inside a superconductor, the *phase* around the loop must be a multiple of 2π.

In a region of the superconductor where the density of superconducting electrons is a constant, and there is zero current flow, then statement (ii) implies

$$\int j \cdot dl = \frac{q}{m}(\hbar \delta S_{loop} - q \int A \cdot dl) = 0 \tag{28}$$

where we used Eq. (11), $j_s = \frac{q}{m}|\psi|^2(\hbar\nabla S - qA)$ for the current. U sing Stokes theorem to change the path integral to a surface integral yields,

$$\int A \cdot dl = \int (\nabla \wedge A) \cdot da = \int B \cdot da = \phi = \frac{\hbar}{q} \int \nabla S \cdot dl = \frac{\hbar}{q} 2\pi n = n\phi_0 \tag{29}$$

where $\phi_0 = h/q$. In all superconductors found so far, $q = 2e$ indicating electron pairing as the mechanism for formation of Bosons and hence the condensation mechanism. The flux quantum in superconductors is

$$\phi_0 = \frac{h}{2e}; \qquad \text{flux quantum} \tag{30}$$

This is the smallest amount of flux that can enter a superconductor where the fundamental Bosons have charge $2e$. Its value has been confirmed in all known superconductors to very high precision. It is clear that the flux quantum provides confirmation of the electron pairing theory of superconductivity. Manipulation of flux quanta is also being considered for new applications such as information storage and quantum computing. Note Another way to look at the flux quantum is to consider a momentum wavefunction,

$$e^{ipx/\hbar} \to e^{i(p-qA)/\hbar} \tag{31}$$

Now consider the change in the wavefunction in going around a loop,

$$\delta\psi = Exp[-i\frac{q}{\hbar} \int A \cdot dl] \tag{32}$$

We then require that the phase change is $2\pi n$, which recovers the results above.

G. An isolated vortex in a superconductor

Before discussing the behavior of superconductors as a function of magnetic field and temperature we consider the structure of vortices, within London theory.

We use London theory as it is not possible to solve explicitly for the behavior of the order parameter and magnetic field near a tube of normal material, at least not the full analytic solution using the GL equations. However a great deal of understanding is derived from the solution which can be found explicitly in the limit $\lambda/\xi >> 1$, which is the London limit. In the limit $\lambda >> \xi, \lambda - \xi \to \lambda$, so it is evident that the energy gain due to flux penetration dominates the energy cost due to loss of the condensation energy. We start with the London equation,

$$\nabla^2 B - \frac{B}{\lambda^2} = -\frac{\phi_0}{\lambda^2}\delta(\vec{r}), \quad \text{where} \quad \lambda = (\frac{m}{\mu_0 q^2 |\psi|^2})^{1/2} \tag{33}$$

To model a vortex using this equation, we assume that the vortex core is small and can be approximated by a delta function. This is treated as a boundary condition at the origin in a cylindrical co-ordinate system. This approach captures the way in which a magnetic field penetrates from the vortex core and also the current circulation around the core. In cylindrical co-ordinates the radial part of Eq. (33) becomes,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial B}{\partial r}) - \frac{B}{\lambda^2} = -\frac{\phi_0}{\lambda^2}\delta(r), \tag{34}$$

where r lies in the x-y plane. The solution to this equation is a Bessel function,

$$B(r) = B_0 K_0(\frac{r}{\lambda}) \quad \text{in the z direction} \tag{35}$$

The Bessel function $K_0(x)$ has the following behaviors:

$$K_0(x \to 0) \sim Ln(1/x) \; ; \quad K_0(x \to \infty) \sim e^{-x} \tag{36}$$

It is clear that this is unphysical as $x \to 0$, but this is reasonable as this theory is invalid for distances $r < \xi$. It is meant to describe behavior on length scales $r >> \xi$. For many type II superconductors the London approximation is very good because $\lambda >> \xi$. Current circulation is induced by the magnetic field gradient according to the Maxwell equation $\mu_0 J = \nabla \wedge B$, this yields,

$$J(r) = J_0 K_1(\frac{r}{\lambda}) \qquad \text{in the } \theta \text{ direction} \tag{37}$$

This Bessel function is related to $K_0(x)$ via, $K_1(x) = -dK_0(x)/dx$. It has the following limiting behaviors,

$$K_1(x \to 0) \sim \frac{1}{x} \; ; \qquad K_1(x \to \infty) \sim e^{-x} \tag{38}$$

At distances $r < \lambda$ vortices in superconductors look a lot like vortices in superfluid Helium II. However at long distances $r > \lambda$ they are screened and the magnetic field and current decay exponentially.

There is only one parameter remaining in the construction above, and that is the magnetic field B_0. This is set by the requirement that the flux be quantized, that is,

$$\int_0^\infty B_0 K_0(\frac{r}{\lambda}) 2\pi r dr = \phi_0 \tag{39}$$

where $\phi_0 = h/q$ is the flux quantum. This is a tabulated integral (e.g. Mathematica can do it),

$$\int_0^\infty x K_0(x) dx = 1 \tag{40}$$

Solving for B_0 yields

$$B_0 = \frac{\phi_0}{2\pi\lambda^2} \;\text{; which implies}\quad J_0 = \frac{\phi_0}{2\pi\mu_0\lambda^3} \tag{41}$$

The *Helmholtz free energy*(per unit length) of an isolated vortex is given by,

$$\epsilon_1 = \frac{1}{2\mu_0} \int_0^\infty (B^2 + \lambda^2 \mu_0^2 J^2) 2\pi r dr \tag{42}$$

which explicitly shows the contributions of the field and the current. In the large λ/ξ limit, this is dominated by the regime $\xi \le r \le \lambda$, so we find an approximate value of the vortex energy by using,

$$\epsilon_1 \approx \frac{1}{2\mu_0} (\frac{\phi_0}{2\pi\lambda^2})^2 2\pi \int_\xi^\lambda [r(Ln(\frac{\lambda}{r}))^2 + r(\frac{\lambda}{r})^2] dr$$

$$= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \int_{\xi/\lambda}^1 [x(Ln(x))^2 + \frac{1}{x}] dx$$

$$= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} [\frac{x^2}{2}(\frac{1}{2} - Ln(x) + Ln(x)^2) + Ln(x)]|_{\frac{\xi}{\lambda}}^1$$

$$\approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} Ln(\frac{\lambda}{\xi}) \tag{43}$$

Notice that the energy cost of forming the vortex is dominated by the kinetic energy of the superconducting electrons (the logarithmic term). The energy cost due to the magnetic field is relatively small. In addition we should add the energy cost of the normal core of the vortex. This energy is

$$\epsilon_{core} \approx \pi\xi^2 \frac{N(\epsilon_F)\Delta^2}{2} = \pi\xi^2 b \frac{a^2}{2b^2} = \pi\xi^2 b \frac{|\psi|^2}{2} = \pi\xi^2 \frac{mb}{\mu_0 q^2 \lambda^2} \tag{44}$$

where the last expression is found using Eq. (5). For large ratios of λ/ξ this core energy is also relatively small compared to the kinetic energy of Eq. (44). However as we shall see later the core energy is very important in pinning of vortices.

H. The lower critical field H_{c1}

The difference in Gibb's free energy between a superconductor containing no vortices and a superconductor containing one vortex, for a field parallel to a thick slab in the large λ/ξ limit, is given by,

$$\delta g_{GL} = \epsilon_1 - \int_0^\infty 2\pi r dr B \cdot H = \epsilon_1 - \phi_0 H \tag{45}$$

The lower critical field is determined by when $\delta g_{GL} = 0$. For fields higher than this, it is favorable for flux to enter the superconductor while for fields lower than this, the Meissner state is favored. In the limit of large λ/ξ, we can use the vortex energy given in Eq. (44) so that,

$$H_{c1} = \frac{\epsilon_1}{\phi_0} \approx \frac{\phi_0}{4\pi\mu_0\lambda^2} Ln(\frac{\lambda}{\xi}) \tag{46}$$

Just above H_{c1} there is a rapid influx of vortices as the interaction between vortices is relatively weak until their separation is less than λ. The vortices also pack efficiently to maximize the decrease in magnetization, which favors the triangular stacking of vortices (this is typical for central force systems).

I. The upper critical field H_{c2}

At the upper critical field we can make several approximations which make the LG equations much simpler. Firstly the order parameter ψ is small so we can ignore the non-linear term. Secondly, we can assume that the vector potential inside the superconductor is nearly at the value specified by the external field, e.g. $A = (0, \mu_0 Hx, 0)$ we shall use this choice of potential for convenience though other choices which satisfy $\nabla \cdot A = 0$ are equally valid due to gauge invariance. With these approximations and assumptions, the LG equation reduces to,

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2m}(-i\hbar\frac{\partial}{\partial y} - q\mu_0 Hx)^2\psi - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} = -a\psi \tag{47}$$

This is the Schrödinger equation for a particle in a magnetic field in the z-direction. The energy eigenvalues are known to be,

$$|a| = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m} \tag{48}$$

where ω_c is the cyclotron frequency,

$$\omega_c = \frac{q\mu_0 H}{m} \tag{49}$$

However we must interpret Eq. (48) in an unusual way. We know $|a|$ and we must find the *largest* applied field H that corresponds to it. The largest field occurs when $k_z = 0$ and $n = 0$, so we have the simple result (using (49) in (48), with $k_z = n = 0$),

$$H_{c2} = \frac{2m|a|}{\hbar\mu_0 q} = \frac{\phi_0}{2\pi\mu_0\xi^2} \tag{50}$$

where we have used $\xi^2 = \hbar^2/(2m|a|)$ to find the last expression in Eq. (50). Since $|a| \sim |T - T_c|$, the critical field, H_{c2} approaches zero linearly near the critical temperature. Similarly, the lower critical field also approaches zero linearly near the critical temperature. Finally we have the interesting result,

$$\frac{H_{c2}}{H_{c1}} = 2(\frac{\lambda}{\xi})^2 \frac{1}{Ln(\lambda/\xi)} \tag{51}$$

This expression demonstrates that even for moderate values of λ/ξ, the two critical fields H_{c2} and H_{c1} are well separated.

II. SCALING THEORY

Though finding exact critical exponents is difficult, scaling theory provides exact relations between critical exponents providing methods to check behaviors calculated in different ways. First we go through the scaling theory of magnetic phase transitions. We then extend the analysis to consider scaling under changes in length.

A. Scaling theory of Ising phase transitions

The objective of the analysis is to find relations between the critical exponents $\alpha, \beta, \delta, \gamma, \eta, \nu$ that control behavior near the Ising critical point. We use the definitions,

$$\beta H = K \sum_{ij} S_i S_j + h \sum_i S_i; \quad M \sim \frac{\partial F}{\partial h}; \quad \chi \sim \frac{\partial M}{\partial h} \tag{52}$$

We also define the correlation function,

$$C(r) =< S(0)S(r) > - < S(0) >< S(r) >; \quad \text{and} \quad \chi \sim \int dV C(r) \tag{53}$$

Now we assume that the correlation length is the key quantity in the scaling theory so that the scaling behavior is of the form,

$$F(T,h) = t^{2-\alpha} F_s(h\xi^y); \quad M(T,h) = t^{\beta} M_s(h\xi^y); \quad \chi(T,h) = t^{-\gamma}\chi_s(h\xi^y); \quad C(r) = r^{-p} C_s(r/\xi, h\xi^y) \tag{54}$$

where $t = |T - T_c|$, and $y > 0$. We also define $\xi^y = t^{-\Delta}$, so that $\nu y = \Delta$, where Δ is the gap exponent. We also have $p = d - 2 + \eta$, and $\xi = t^{-\nu}$. The scaling functions have the property that as their argument $x = h\xi^y = h/t^{\Delta}$ goes to zero, the scaling functions must approach a constant. Moreover the scaling assumption states that for $h < \xi^{-y}$ the scaling functions are constant. Moreover, as $x \to \infty$, the scaling functions go to zero. First consider the behavior of the magnetization when we are at the critical point, so that,

$$M(t = 0, h \neq 0) \sim t^{\beta} M_s(x \to \infty) \sim h^{1/\delta}; \quad \text{so that} \quad M_s(x) \sim x^k \tag{55}$$

where,

$$t^{\beta} x^k = t^{\beta} (\frac{h}{t^{\Delta}})^k = h^{1/\delta}; \quad \text{so that} \quad k = 1/\delta; \quad \text{and} \quad \Delta = \beta\delta \tag{56}$$

Now consider the relation between the magnetization and the susceptibility,

$$M \sim \int \chi dh \sim \int_0^{\xi^{-y}} \sim t^{-\gamma} t^{\Delta} \sim t^{\beta}; \quad \text{so that} \quad \beta = \Delta - \gamma \tag{57}$$

In a similar manner,

$$F \sim \int M dh \sim \int_0^{\xi^{-y}} t^{\beta} t^{\Delta} \sim t^{2-\alpha}; \quad \text{so that} \quad \beta + \Delta = 2 - \alpha \tag{58}$$

Finally, consider the scaling of the correlation function in the case where $h\xi^y$ is zero, so that C_s is a constant for $r < \xi$ and zero otherwise. We then have,

$$\chi \sim \int d^3 r C(r) \sim \int_a^{\xi} dr r^{d-1} r^{-p} C_s(r/\xi, h\xi^y) \sim \xi^{d-(d-2+\eta)} \sim t^{-\gamma}; \quad \text{so that} \quad \gamma = \nu(2-\eta) \tag{59}$$

These exponent relations are usually written in the form,

$$\Delta = \beta + \gamma; \quad \gamma = \nu(2-\eta) \ \ (Fisher); \quad \alpha + 2\beta + \gamma = 2 \ \ (Rushbrooke); \quad \gamma = \beta(\delta - 1) \ \ (Widom) \tag{60}$$

Since we have added the "gap" exponent Δ, there are seven exponents in the problem. We have four exponent relations so that only three exponents are independent. Josephson introduced another relation, called the hyperscaling relation. He introduced the hypothesis that the singular part of the free energy scales as $1/\xi^d$. This implies that,

$$f_{sing} \approx \xi^{-d} \approx t^{2-\alpha}; \quad \text{so that} \quad d\nu = 2 - \alpha \ \ (Josephson, \ or \ hyperscaling \ relation) \tag{61}$$

The hyperscaling relation is considered the most likely of the scaling relations to fail and for example is known to fail in some heterogeneous models such as the Spin glass model.

These exponent relations extend to the liquid gas phase transition and to many other problems that have more complex order parameters, such as superconductivity and $O(n)$ magnets. If there are more parameters in the problem that must be tuned to find the critical point, then it may be necessary to extend the model to a system with three independent exponents.

B. Generalized scaling relations, finite size scaling

In the renormalization group theory and in the analysis of results of simulations and experiments, it is interesting to consider the change in properties under rescaling by a length b. Using the results of the previous section and use the relation between parameter variation and length scale $\xi = t^{-\nu}$, we may write Eq. () as,

$$M(t,h) = b^{-\beta/\nu} M_s(hb^{D_h}, tb^{D_t}); \qquad \chi(t,h) = b^{\gamma/\nu}\chi_s((hb^{D_h}, tb^{D_t}); \qquad C(r) = b^{-p}C_s(r/b, hb^{D_h}, tb^{D_t})) \tag{62}$$

The length scale b does not have to be the correlation length, so for example in a finite system of size L, we immediately find the behavior of the system as the size of the sample increases. e.g. the magnetization at $h = 0, t = 0$, $M \sim L^{-\beta/\nu}$, which is the finite size scaling behavior of the magnetization at the critical point. Comparing this formulation with the formulation above, we find that,

$$D_t = 1/\nu; \qquad D_h = \Delta/\nu \tag{63}$$

As we shall see later the renormalization group finds the exponents D_t and D_h.

C. Lower critical dimension

The lower critical dimension is the dimension below which thermal fluctuations are always relevant. In english that means thermal fluctuations are strong at any temperature and they destroy long range order. For the fluctuations to destroy long range order of, for example, a ferromagnet, large scale fluctuations must have finite energy. We can find the typical energy of a long range fluctuation by considering a domain wall. First consider an Ising model where a domain wall consists of an interface between an up spin half-space and a down spin half-space. It is easy to calculate the enery of this interface (at zero temperature), and we write,

$$E_{interface} = 2JL^{d-1}; \qquad \text{Ising domain wall, so} \quad d_{lc} = 1 \tag{64}$$

From this expression it is clear that for $d = 1$ the domain wall energy is finite so that thermal fluctuations destroy long range order at any finite temperature. However for any $d > 1$, the interface energy grows with the size of the domain wall, so the ordered state is stable for small but finite temperature. At high enough temperature order is destroyed because the surface tension goes to zero. This low critical dimension also applies to the liquid-gas phase transition.

Now consider a superconductor where the order parameter has a phase degree of freedom. This enables the domain wall energy to be reduced. In a system of size L, the domain wall width is L instead of 1 as occurs in the ising case. The simplest model to illustrate this behavior is a spin model where the spin can rotate with one angular degree of freedom. In that case,

$$H = \sum_{ij} J_{ij}\vec{S}_i \cdot \vec{S}_j = \sum_{ij} J_{ij}|S|^2 Cos(\theta_{ij}) \tag{65}$$

where θ_{ij} is the angle between the two spins. If we make a domain wall of width l in this model, the angle between adjacent spins is π/l, so the energy of the domain wall is,

$$E_{interface} = 2JL^{d-1}l(cos(\pi/l) - 1) \approx 2\pi^2 J\frac{L^{d-1}}{l} \to 2\pi^2 JL^{d-2}; \qquad \text{so}, \quad d_{lc} = 2 \tag{66}$$

where the last expression is found by setting $l \to L$. This shows that two dimensional superconductors are unstable to domain formation. This is similar to what we found for the Bose gas, where there is no true Bose condensation in two dimensions, however the physical origin of the two effects is different. In the limit where the thickness of a sample is of order or less than the coherence length, we expect strong fluctuations in superconducting domains due to this effect.

D. Upper critical dimension - Lifshitz criterion

Below the lower critical dimension, no finite temperature phase transition occurs. Nevertheless there are sometimes interesting behaviors as $T \to 0$, especially in quantum systems where quantum critical points may occur at zero temperature.

As the spatial dimension increases, the fluctuations become less important due to the higher connectivity of the systems. The upper critical dimension is the the dimension above which fluctuations have no effect on the critical exponents. They may still change non-universal properties such as the critical temperature, however they do not alter the leading order critical exponents. This means that above the upper critical dimension mean field theory is correct.

Lifshitz considered the ratio, $\frac{C(\xi)}{m^2}$ which compares the fluctuations to the order parameter squared. If this ratio goes to zero as we approach the critical point, then fluctuations are irrelevant. Carrying this through we find that,

$$\frac{\xi^{-p}}{m^2} \approx t^{(d-2+\eta)\nu - 2\beta} \tag{67}$$

To find the critical dimension, we use the mean field values $\beta = 1/2, \nu = 1/2$, to find that,

$$(d-2+\eta)\nu - 2\beta = 0 \qquad \to \qquad d_{uc} = 4 \tag{68}$$

The upper critical dimension is then four, and below that value the fluctuations modify the critical exponents. Note that the critical dimension for a tricritical point is different and there are other cases where $d = 4$ is not correct. However for superconductors, liquid-gas transitions and homogeneous magnets it is four.

From the discussion of upper and lower critical dimensions it is evident that we happen to live in the window of dimensions where fluctuations are relevant. In many ways three dimensions is the most interesting and complex dimension for critical phenomena, at least for the models we have discussed in this course.

The above exponent relations and critical dimensions are EXACT, which is surprising given the simplicity of the analysis. Finding the values of the two remaining unknown exponents for $d_l < d < d_u$ is much more challenging and lead to the development of many different tools and approaches, including the renormalization group, series expansions and high precision computational methods.

E. The renormalization group for Ising systems

The renormalization group consists of analytic and computational schemes to integrate systematically over degrees of freedom in a system near a critical point. After integration, the control parameters, for example temperature and magnetic field in a magnetic system, are rescaled to restore the system Hamiltonian to its original form. The behavior of the control parameters under this rescaling enable calculation of the critical behavior of the model.

F. Exact RG in one dimension, Midgal-Kadanoff in arbitrary dimension

First we discuss the most straighforward approach, called decimation, where the length rescaling is achieved by integrating of a subset of the spins in the system. This can be carried out exactly in one dimension where we consider,

$$-\beta H = K \sum_i S_i S_{i+1}; \qquad \text{so} \quad Z = \sum_{\{S_i\}} e^{K \sum_i S_i S_{i+1}} \tag{69}$$

Now we decimate by summing only over spins that are on even sites of the lattice, to find,

$$Z = \sum_{\{S_i\}}^{i\ odd} \prod_{i\ odd} 2Cosh(S_i + S_{i+2}) \tag{70}$$

Now note that,

$$Cosh(S_i + S_{i+2}) = e^{KS_i} e^{KS_{i+2}} + e^{-KS_i} e^{-KS_{i+2}} = Cosh^2 K[(1 + vS_i)(1 + vS_{i+2}) + (1 - vS_i)(1 - vS_{i+2})] \tag{71}$$

where we used the identity, $e^{\alpha S} = Cosh(\alpha) + SSinh(\alpha)$. Expanding and then using this identity again we find,

$$Cosh(S_i + S_{i+2}) = Cosh^2 K[1 + v^2 S_i S_{i+2}] = e^{K' S_i S_{i+2}} \tag{72}$$

where the renormalization of the coupling constants is,

$$v' = v^2; \qquad \text{with} \qquad v = tanh(K); \qquad \text{RG equation} \tag{73}$$

This "decimation" process then recovers the original Hamiltonian with a renormalized coupling constant K'. We look for fixed points of the iteration of this renormalization group equation leads to two fixed points $v* = 0,\ 1$, corresponding to zero and infinite temperature. The infinite temperature fixed point is the attractive one, indicating that there is no long range order at any finite temperature.

Now we consider the Ising model on a hypercubic lattice. Hypercubic lattices are bipartite, so we can sum over one sublattice to find a candidate reduced partition function, however it is not possible to reduce the partition function to the original form, so it is not possible to find the exact renormalization group equations. However a simple approximation can be used to enable an analytic approximation, called the Migdal-Kadanoff approximation. In this approximation, bonds are moved so that the remaining problem consists of double bonded, doubly connected sites on one sublattice. We can then sum over the sublattice, and the only change to the one dimensional solution is that the renormalization group equation becomes,

$$tanh(K') = tanh^2(dK) \tag{74}$$

where d is the dimension. This equation has three fixed points $K^* = 0, 1, K_c(d)$, where the non-trivial fixed point $K_c(d)$ has values $K_c(d = 2) = 0.305$ and $K_c(3) = 0.121$. Due to the scaling form of the free energy, the behavior of the RG equations near the fixed point are,

$$K'(K) = b^{D_t}|K^* - K|; \qquad \text{or} \qquad v'(v) = b^{D_t}|v^* - v| \tag{75}$$

where $v = tanh(K)$ and in our decimation procedure $b = 2$. We carry through this analysis in two dimensions. In terms of v the RG equation in two dimensions is,

$$v' = \frac{4v^2}{(1+v^2)^2}; \qquad \text{where we used} \qquad tanh(2x) = \frac{2tanh(x)}{1+tanh^2(x)} \tag{76}$$

Now we expand near the critical point,

$$v' = v^* + x'; \quad v = v^* + x; \qquad \text{so that} \qquad v^* + x' = \frac{4(v^*+x)^2}{(1+(v^*+x)^2)^2} \tag{77}$$

Expanding and keeping linear terms in x gives,

$$v^* + x' = \frac{4(v^*)^2}{(1+(v^*)^2)^2} + \frac{8(v^* - (v^*)^3)}{(1+(v^*)^2)^3}x \tag{78}$$

so that,

$$x' = \frac{8(v^* - (v^*)^3)}{(1+(v^*)^2)^3}x = 2^{D_t}x \tag{79}$$

Using $v^* = Tanh(K^*)$, with $K^* = 0.305$ and solving for D_t yields,

$$D_t = \frac{ln(1.679)}{ln(2)} = .747 = 1/\nu \tag{80}$$

This gives a value of around $\nu \approx 1.32$. Recall that the mean field value is $\nu = 1/2$, while the exact value in two dimensions is $\nu = 1$. The value extracted from this simple RG is thus quite promising.

We would like to find a two parameter RG so that we can find both exponents D_t and D_h, however that is quite a long calculation even within the Migdal-Kadanoff approximation, with the result for two dimensions,

$$K' = \frac{1}{2}ln(Cosh(4K)); \qquad h' = h[1 + tanh(4K)] + O(h^2) \tag{81}$$

The fixed point $K^* = 0.305, h^* = 0$. A linear expansions leads to the same results as above for the thermal exponent and for the magnetic field rescaling,

$$h' = 1.84h = 2^{D_h}h, \qquad \text{so that} \quad D_h = 0.88 \tag{82}$$

where D_h is related to the other scaling exponents through, $D_h = \Delta D_t$. Subsequently it has been shown that the Migdal-Kadanoff bond moving procedure leads to results that are equivalent to solving the Ising model on a fractal/hierarchical lattice.

G. Field theory formulation of RG

Wilson introduced the perturbative RG with Hamiltonian,

$$\beta H = \beta H_0 + U; \quad where; \quad \beta H_0 = \int d^d r[\frac{t}{2}m^2 + \frac{K}{2}(\nabla m)^2]; \quad U = u\int d^d r\ m^4 \tag{83}$$

It is more convenient to work with the Fourier transforms,

$$\beta H_0 = \int \frac{d^d q}{(2\pi)^d}\frac{1}{2}(t + K\vec{q}^2)|m(\vec{q})|^2; \quad U = u\int \frac{d^d q_1}{(2\pi)^d}\int \frac{d^d q_2}{(2\pi)^d}\int \frac{d^d q_3}{(2\pi)^d} m(\vec{q}_1)m(\vec{q}_2)m(\vec{q}_3)m(-\vec{q}_1 - \vec{q}_2 - \vec{q}_3) \tag{84}$$

The momentum space integrals have upper limit Γ, and we introduce a rescaling parameter b, so that $\Lambda' = \Lambda/b$. The RG procedure is then to integrate over momenta in the range $\Lambda' < q < \Lambda$, and then restore the Hamiltonian to its original form, and restore the upper limit to its initial value Λ. The value of Λ is related to $1/a$ where a is the lattice spacing of the Ising lattice. The RG procedure then integrates over short length scales and finds the way that the parameters in the Hamiltonian scale when this integration is carried out. This procedure cannot be carried out exactly, so they are approximated by a pertubation of the quantity U using a cummulant expansion

$$ln < e^{-U} >_{H_0} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} << U^n >>; \qquad \text{where} \qquad << U >>=< U >; \; << U^{n>1} >>=< (U- < U >)^n > \tag{85}$$

The cummulants, or irreducible moments, are of interest not only here but in general since they are extensive quantities. This is not true of the ordinary moments. In calculating the cummulants we are calculating expectations values in a Gaussian measure, so we can use Wick's theorem, which states that the average of a correlation function of order m is zero for m odd, and for m even it is equal to a sum over the values of all pair contractions of the operators. The number of such contractions is $n!/[2^{n/2}(n/2)!]$. Those interested in the technical details of carrying out this calculation can consult many different sources, with a very good one being Daniel Amit, "Field theory, the renormalization group, and critical phenomena".

If this perturbative procedure is carried out to two loops, it leads to the RG equations,

$$\frac{dt}{dl} = 2t + \frac{4(n+2)K_d\Lambda^d}{t + K\Lambda^2}u; \qquad \frac{du}{dl} = (4-d)u + \frac{4(n+8)K_d\Lambda^d}{(t + K\lambda^2)^2}u^2 \tag{86}$$

If $u = 0$, the model is called the Gaussian model and for $T > T_c$ or $t > 0$, it has behavior like that of mean field theory. However for $t < 0$ it is unphysical as the integrals diverge. When u is finite, there is a new fixed point that is found by solving the above equations to find,

$$u^* = \frac{(t + K\Lambda^2)^2}{4(n+8)K_d\Lambda^d}\epsilon = \frac{K^2}{4(n+8)K_4}\epsilon + O(\epsilon^2); \qquad t^* = -\frac{2u*(n+2)K_d\Lambda^d}{t + K\Lambda^2} = -\frac{n+2}{2(n+8)}K\Lambda^2\epsilon + O(\epsilon)^2 \tag{87}$$

Carrying out a linear expansion near the fixed point yields,

$$D_t = 2 - \frac{n+2}{n+8}\epsilon; \qquad D_u = -\epsilon \tag{88}$$

where $\epsilon = 4 - d$. In four dimensions $\epsilon = 0$, so that $D_t = 2 = 1/\nu$ so $\nu = 1/2$ as expected for mean field theory. For ϵ finite, the value of n is important, with different values of n corresponding to different systems, including: $n = 1$ for Ising and liquid gas transitions; $n = 2$ for superfluid/superconductor and XY systems; $n = 3$ for Heisenberg models etc. For Ising systems in three dimensions $n = 1, \epsilon = 1$, so that $D_t = 5/3 = 1/\nu$, so that $\nu = 0.6$. The correct value in three dimensions is $\nu = 0.63$. Higher order expansions of the field theory get close to this value, however the most accurate values of the exponents are found using computational methods such as Monte Carlo methods. For the XY (n=2) model in three dimensions the above predicts that $\nu = 5/8 = 0.625$, while the correct value is 0.671.

Though the calculation of critical exponents is an important success of RG theory, its greatest success is the prediction of the general behavior of models, in particular in helping decide which operators or physical effects are relevant and which are not for the general topology of the phase diagrams.

III. DYNAMICS OF MANY PARTICLE SYSTEMS

In the last three lectures we will discuss a few key issues in the dynamics of many particle systems. This is a very broad subject however some of the tools developed for equilibrium phenomena can be applied very successfully to the study of dynamics. We consider a couple of examples.

A. Dynamics of phase transitions

In this area there are again many universal features and concepts so we can discuss the Ising model first and then discuss how the results extend or transfer to other problems. Many different dynamical questions can be asked but in general dynamics can be considered to be "close to equilibrium" and dynamics far from equilibrium. An example of dynamics close to equilibrium is the relaxation of ferromagnet when the temperature is changed by a small amount. An example of dynamics far from equilibrium is when we have a ferromagnet in the "up" magnetized state and we apply a small "down" magnetic field. In that case we can ask how the system goes from the completely wrong "up" state to the correct "down" state. There are also collective dynamical states, such as self-organized critical states, that do not exist in the absence of a flow. We shall talk about these systems later.

1. *Non-conserved and conserved dynamics driven by domain wall free energy*

A variety of different dynamical processes can lead to the same equilibrium state, however the rate at which equilibrium is approached depends strongly on the type of dynamics. There are two basic classes of dynamics of phase transitions though there are many subclasses. Ginzburg-Landau mean field theory may be extended to treat both conserved and non-conserved dynamics of the order parameter characterizing a system. In the case of a ferromagnet the dynamical GL equations describe relaxation of the magnetization, while in a superconductor they describe the relaxation of the order parameter magnitude and phase. The simplest example to consider is relaxation of a surface in one dimension where the height of the surface is characterized by $h(x)$. This can describe for example the interface between an up spin domain and down spin domain in a ferromagnet. We consider first a continuum model to describe the relaxation of a perturbation of a domain wall in two dimensional Ising ferromagnet. We assume that energy of the domain wall depends on its length times its energy per unit length, so that,

$$E_{DW} = \sigma \int dx \sqrt{1 + (\frac{dh}{dx})^2} \approx constant + \frac{\sigma}{2} \int dx (\frac{dh}{dx})^2 \tag{89}$$

where the last expression is found by making the small angle approximation to expand the square root to leading order. Here σ is the surface tension. We assume that it is isotropic, which is only true close to the critical temperature. For a ferromagnetic Ising model at low temperature $\sigma \approx 2J$. The equilibrium state is found by doing a variation of the domain wall energy using the Euler-Lagrange equations. To study relaxation processes, we consider two cases. The simplest case, called Allen-Cahn theory, is to assume a simple non-conserved relaxation dynamics where,

$$\frac{\partial h}{\partial t} = \gamma \mu = \gamma \frac{\partial E_{DW}}{\delta h} = \gamma \sigma \frac{\partial^2 h}{\partial x^2} \tag{90}$$

where we introduced the rate γ, and chemical potential μ. This is a course grained approach to dynamics in the same sense that GL theory is a coarse grained approach to equilibrium phase transitions. In fact, later we shall extend the GL model to include dynamics of this type. This is the diffusion equation, so we expect that length and time are related through $x \propto t^{1/2}$. A general scaling approach to partial differential equations illustrates this scaling. We assume the form,

$$h(x,t) = t^{-a} f(x/t^b) = t^{-a} f(s); \qquad \text{where} \qquad s = x/t^b \tag{91}$$

Substituting this expression into the diffusion equation gives,

$$-at^{-a-1} f(s) - bxt^{-a-b-1} f'(s) = -at^{-a-1} f(s) - bt^{-a-1} s f'(s) = \gamma \sigma t^{-a-2b} f''(s) \tag{92}$$

We now choose $b = 1/2$ so that the time dependence drops out of the equation. This proves that all solutions of this form have the relationship $x \sim t^{1/2}$ where $f(s)$ is the scaling function or shape function which satisfies,

$$-af(s) - bsf'(s) = \gamma \sigma f''(s) \tag{93}$$

This type of dynamics is typical of Ising domain walls where the number of up and down spins are not conserved.

Now consider a converved domain wall dynamics which is often called Cahn-Hilliard theory, where the number of particles is conserved. In this case the relaxation occurs by transport of particles along the surface through a surface current (Fick's law,

$$\vec{j} = -k \nabla \mu \rightarrow j = -k \frac{\partial \mu}{\partial x} \tag{94}$$

where k is a constant. The continuity equation ensures conservation of particle number, so that,

$$\frac{\partial h}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{95}$$

so that for a one dimensional interface,

$$\frac{\partial h}{\partial t} = -\gamma_1 k \frac{\partial^4 h}{\partial x^4} \tag{96}$$

A similarity analysis of this equation gives,

$$-at^{-a-1}f(s) - bxt^{-a-b-1}f'(s) = -at^{-a-1}f(s) - bt^{-a-1}sf'(s) = -\gamma_1 kt^{-a-4b}f''''(s) \tag{97}$$

We now choose $b = 1/4$ so that the time dependence drops out of the equation. This proves that all solutions of this form have the relationship $x \sim t^{1/4}$ where $f(s)$ is the scaling function or shape function which satisfies,

$$-af(s) - bsf'(s) = -\gamma_1 kf'''' \tag{98}$$

Clearly the dynamics of the convserved case is much slower due to the requirement that transport is only along the surface, instead of through exchange with a reservoir.

IV. DYNAMICS OF FIRST ORDER PHASE TRANSITIONS

We will look at two perspectives on the dynamics of first order phase transitions. The mean field perspective based on Landau theory, and the critical droplet theory that takes spatial dimension into account. First we look at the critical droplet theory.

Consider an ising magnet in the up spin state for temperatures $T < T_c$. Now consider applying a small magnetic field that favors the down spin state. The energy of a droplet of size R is approximately,

$$E_{droplet} = -2hc_d R^d + 2\sigma k_d R^{d-1} \tag{99}$$

where c_d, k_d are constants, e.g. for two dimensions $s_d = \pi, k_d = 2\pi$. The field energy is minimized in the down spin state, but to create a droplet we need to create a domain wall which costs energy proportional to the surface area. If we minimize this expression with respect to R, we find the critical droplet size,

$$R_c = \frac{(d-1)k_d\sigma}{dc_d h}, \qquad \text{and} \qquad E_{barrier} = E_{droplet}(R_c) = -2hc_d(\frac{(d-1)k_d\sigma}{dc_d h})^d + 2\sigma k_d(\frac{(d-1)\sigma k_d}{dc_d h})^{d-1} \tag{100}$$

For the case of a small field h, the energy barrier is large, so the rate at which critical droplets are nucleated is low. As the field increases, or as σ decreases, the nucleation rate increases. At low temperatures and fields the nucleation rate is exponentially activated and is proportional to $e^{-E_{barrier}/(k_B T)}$.

Now we consider the behavior of a droplet of down spins in background of up spins. First consider the case of zero applied field. Droplets or clusters are generated randomly and if, through a rare event, a droplet of size $R >> \xi$ is generated randomly, it will relax to equilibrium through domain wall driven NCOP dynamics, so that,

$$\frac{dR}{dt} = -\gamma\frac{\sigma}{R}; \qquad \text{so} \qquad R(t) = c(t - t_0)^{1/2} \tag{101}$$

where $1/R$ is the curvature. If a field is applied to the system and a rare thermal fluctuation creates a down cluster with $R > R_c$, the radius of the droplet will increase according to,

$$\frac{dR}{dt} = -\gamma\frac{\sigma}{R} + \gamma h \tag{102}$$

At long times the first term can be ignored and solution to the remaining equation shows that the droplet grows linearly with time $R \sim \gamma ht$. In many experiments where nucleation is slow, the growth time is much shorter then the nucleation time, so the growth is studied using a nucleation center that is placed in the system to seed the growth. A practical example of nucleation seeding is cloud seeding which works by taking advantage of the fact that many clouds contain supercooled water vapor. Despite the supercooling, the critical water droplet size is quite large and there are relatively few natural nucleation centers in clouds. Cloud seeding provides nucleation centers for the formation of liquid water leading to the increased possibility of rain.

A. Spinodal decomposition

In our example of an up spin configuration at $T < T_c$ in an applied field that favors the down state, we can consider increasing the field until no energy barrier to formation of the down spin state remains. The field at which this occurs is called the spinodal point. Within the droplet theory, the spinodal line is defined by the condition at which the energy barrier to formation of the stable phase from the metastable phase goes to zero. For fields below this point we are in the metastable regime, while for fields larger than this point we are in the unstable regime. A similar subdivision applies to the coexistence regime of the liquid gas system, where the regime closest to the coexistence boundary is metastable, while the other regime is unstable.

Spinodal lines may also be calculated within mean field theory, including the van der Waals theory and Landau theory. Consider the Landau theory for an Ising model in an applied magnetic field,

$$F = \frac{1}{2}a(T-T_c)m^2 + \frac{1}{4}bm^4 - hm \tag{103}$$

The addition of the magnetic field breaks the symmetry between the up and down states. If we choose a positive magnetic field, the energy of the up state (m positive) is lower than the energy of the down state (m negative). At small fields there are still two minima and the down spin configuration is metastable. However we may increase the magnetic field until the down spin configuration is no longer metastable. The point at which the metastability of the down state ceases to exist is the mean field spinodal point. To find this point, we find the extrema for $T < T_c$,

$$bm^3 = h + \alpha m; \qquad \text{where} \quad \alpha = a|T-T_c| \tag{104}$$

The spinodal line is the point at which two of the three solutions to this equation become complex. In a similar way, we may look again at the van der Waals equation, where the spinodal lines are defined by $\partial P/\partial v = 0$, so that,

$$\frac{-k_B T_s}{(v_s-b)^2} + \frac{2a}{v_s^3} = 0 \tag{105}$$

Graphically these points are the peak and trough of the "wiggles" in the van der Waals equation of state. The region between these points is the unstable regime (within mean field theory), while the region between these points and the co-existence line is the metastable regime.

B. Langevin equation and linear response

The Langevin approach reduces the N-particle phase space dynamics to a lower dimensional dynamics. If we consider $M << N$ of the particles in an N dimensional system, we can consider each of the M particles to be essentially independent. Then we are interested in the dynamics of one particle in a reservoir of other particles that are assumed to be in a state of "molecular chaos". In the simplest case, Langevin theory considers the effect of molecular chaos to be a random force, but with the added feature that the perturbations due to the random forces are damped. We then have the basic Langevin equation for one particle in an N-particle system,

$$m\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{B} + \eta'(t) \tag{106}$$

where B is the damping coefficient and $\eta'(t)$ is the random force. The damping coefficient is equal This force is often assumed to be "Gaussian white noise", which means that there are no correlations in the noise, either in time or space. Colored noise introduces correlations in the noise. Note that for conserved dynamics problems the noise must include the effects of the conservation laws. If there is no net direction or flow in the particle dynamics of the system then $< \eta'(t) >= 0$. The Langevin equation provides a coupling between equilibrium properties (the fluctuations in the random force) and non-equilibrium properties (the damping B).

Langevin theory is very nice in that it gives a very direct way to find the relation between the diffusion constant and the damping coefficient. This is demonstrated by solving for the mean square distance that a particle moves, according to the Langevin equation and to calculate a similar quantity from the diffusion equation. The diffusion equation states that,

$$\frac{\partial n}{\partial t} = D\nabla^2 n \quad \text{so that} \quad n = \frac{1}{(4\pi Dt)^{d/2}} e^{-|\vec{r}-\vec{r}_0|^2/4Dt} \tag{107}$$

where n is the single particle density and is normalized to one, d is the spatial dimension, and D is the diffusion constant. From this expression the we find

$$< |\vec{r} - \vec{r}_0|^2 >= 2dDt. \tag{108}$$

Now lets analyse the stochastic Langevin equation. If we take a time average of Eq. (106), we find the simple result,

$$< v(t) >= v(0)e^{-t/\tau} \quad \text{where} \quad \tau = 1/mB \tag{109}$$

is the relaxation time. That is, the relaxation of the velocity to equilibrium is exponential. If there is no net flow, as assumed here, at long times the average velocity is zero in agreement with the MB distribution. With this definition the Langevin equation becomes,

$$\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{\tau} + \eta(t) \quad \text{where} \quad \eta(t) = \eta'(t)/m. \tag{110}$$

We find the mean square distance travelled by the particle using the Langevin equation, by taking a dot product of $\vec{r}$ with the Langevin equation and the averaging over the noise. To carry this out, we use the results,

$$\vec{r}\cdot\vec{v} = \frac{1}{2}\frac{d(\vec{r}\cdot\vec{r})}{dt}; \qquad \vec{r}\cdot\frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d^2(\vec{r}\cdot\vec{r})}{dt^2} - v^2; \qquad < \vec{r}\cdot\eta(t) >= 0.0 \tag{111}$$

to find that,

$$\frac{d^2 < r^2 >}{dt^2} + \frac{1}{\tau}\frac{d < r^2 >}{dt} = 2 < v^2 >= \frac{dk_BT}{m} \tag{112}$$

where the last expression comes from using the equipartition theorem for a monatomic particle system. Solving this equation yields,

$$< r^2 >= \frac{2dk_BT}{M}\tau^2[\frac{t}{\tau} - (1 - e^{-t/\tau})] \quad \text{so as } t \to \infty \qquad < r^2 >= 2dBk_BTt \tag{113}$$

Comparing with the diffusion equation result yields $B = D/k_BT$, which is the Einstein relation for brownian motion. It is one example of a fluctuation-dissipation result where the close to equilibrium response (the damping) is related to the dynamical fluctuations at equilibrium (the diffusion). The damping coefficient B can be measured as the "mobility" defined as the ratio of the terminal velocity over the driving force.

C. Time dependent G-L theory

The Langevin concept of adding a noise term to include the effect of fluctuations is often used to consider the effects of fluctuations on the dynamics of the Ginzburg-Landau theory. To do this the COP and NCOP theories are written for the order parameter, and a noise term is added, leading to,

$$\frac{\partial m}{\partial t} = -\Gamma_1\frac{\delta F}{\delta m} + \eta_N(x,t) \qquad (NCOP); \qquad \frac{\partial m}{\partial t} = \Gamma_2\nabla^2\frac{\delta F}{\delta m} + \eta_C(x,t) \qquad (COP); \tag{114}$$

where F is the G-L free energy of the system (it can be the Gibb's, Helmholtz or other free energy appropriate to the system under study. These are quite general equations where the noise must obey the conservation laws of the system. For the case of Gaussian white noise we have,

$$< \eta_N(x,t)\eta_N(x',t') >= 2k_BT\Gamma_1\delta(x,x')\delta(t,t'); \qquad < \eta_C(x,t)\eta_C(x',t') >= 2k_BT\nabla^2\Gamma_2\delta(x,x')\delta(t,t') \tag{115}$$

V. PERCOLATION - A MODEL WITH $d_u = 6$ AND $\beta_{mf} = 1$

Understanding of continuous or second order phase transitions at equilibrium has lead to a variety of concepts and methods that have been applied to many different problems. One example is the percolation model that provides a very nice connection between fractal scaling and second order phase transitions.

Bernard Julia studied fractals from the point of view of domains of convergence of solutions to equations. He had to do the calculations by hand and published a book in 1918 on the subject. Benoit Mandlebrot brought the subject to a broader community by using computers to generate fractal structures, and collected together many examples of simple rules that can be used to generate fractal sets.

There are many types of fractal objects and many ways that fractals arise in physical processes. One example that we have studied without calling it fractals is the behavior of fluctuating clusters at a second order critical point. Though the RG and scaling theories give us methods for calculating the properties near second order phase transitions, we have not characterized the geometry of the clusters in real space. It turns out the up spin and down spin clusters at the critical point have a distribution of sizes that is a power law, and also the large clusters themselves have a fractal scaling. In this example, the fractal scaling is described by the fractal dimension of a spin cluster through,

$$n_{up}(r) \approx r^{D_f}; \qquad n_{down}(r) \approx r^{D_f} \qquad for \qquad r < R \tag{116}$$

where r is the distance from the center of mass of an up spin cluster and $n_{up}(r)$ is the number of up spins within radius r. For a compact cluster of size R, for $r < R$, we have $D_f = d$ where d is the spatial dimension. For a fractal cluster we have $D_f < d$ which is the characteristic of a fractal object. Fractal objects have holes at all length scales.

A simple geometric model that exhibits many of the features of a phase transition, and in fact can be mapped to a spin model, is the percolation model. This model can be defined on a lattice or graph, but it can also be defined in a continuous medium. For illustration we take the simple example of percolation on a square lattice first. The percolation model considers each of the nearest neighbor bonds of a square lattice to be initially present. It is useful to think of the bonds as resistors or pipes that can carry current or fluid flow. Now consider cutting bonds randomly. What fraction of the bonds must be cut until the square lattice not longer carries current of fluid from top to bottom. This point is called the percolation threshold. We define p to be the fraction of the nearest neighbor bonds that are not cut, so the fraction that are cut is $f = 1 - p$. The order parameter for this problem is derived from a geometric object called the giant cluster, which is largest connected cluster of resistors that remain when a fraction $f = 1 - p$ of the nearest neighbor bonds have been cut. The order parameter is the probability, P_∞, that a bond is part of the giant cluster. The giant cluster is sometimes also called the infinite cluster. Since P_∞ is a probability, it takes a value between zero and one. If none of the bonds in a square lattice are cut, all of the bonds in the lattice are connected so $P_\infty = 1$. When more and more bonds are cut in the square lattice, we eventually reach a point where the network no-longer can carry current or fluid flow across the lattice. The first point where this occurs is the percolation threshold, p_c. At that point, $P_\infty \to 0$. The behavior of P_∞ shows all the scaling properties of an order parameter and near the percolation threshold it goes to zero continuously $P_\infty \sim (p - p_c)^\beta$. However now the infinite cluster is a geometric object so we can make a direct connection to the concept of a fractal dimension. To do this, we write down the scaling law for length rescaling

$$P_\infty(p) = b^{-\beta/\nu} P_s(b^{D_t} p) \to b^{-\beta/\nu} \qquad at\ p_c \tag{117}$$

Scaling theory of second order phase transitions thus predicts that the geometric properties of the infinite cluster are determined by the ratio β/ν, at the critical point. For the percolation model in two dimensions, $\beta = 5/36$, $\nu = 4/3$. Now we can also carry out a fractal analysis of the infinite cluster, so that,

$$n_\infty(b) = b^{D_f} \tag{118}$$

Using the relation,

$$P_\infty = n_\infty / L^d, \qquad \text{we find} \qquad \frac{\beta}{\nu} = d - D_f \tag{119}$$

demonstrating that the fractal dimension of the infinite cluster geometry contains the same information as the scaling exponents near the second order critical point.

An intriguing aspect of the percolation problem is that its mean field behavior is different that the other transitions we have discussed (Ising, liquid-gas, superconductors). In those systems the mean field order parameter exponent is $\beta = 1/2$. The mean field theory of the percolation system is found using a model called the random graph model. In this model, every site may be connected to any other site with probability p. It turns out that the graph constructed in this manner has a tree-like structure near its percolation threshold. For a treelike graph with N sites we can write a recursion relation for the infinite cluster probobality, through,

$$P_{l+1} = \sum_{k=1}^{N} \binom{N}{k} (pP_l)^k (1 - pP_l)^{N-k} = 1 - (1 - pP_l)^N \tag{120}$$

The steady state solution to this equation is,

$$P_\infty = 1 - (1 - pP_\infty)^N \tag{121}$$

Now if the bond probability is taken to be $p = c/N$, we find the simple result,

$$P_\infty = 1 - e^{-cP_\infty} \tag{122}$$

which is the order parameter equation for the mean field model of percolation (e.g. like $m = tanh(\beta Jzm)$) for the Ising model. This is often called the Erdos-Renyi model for a random graph. An interesting feature of this model is that the mean field threshold is at $c = 1$, while the critical exponent is $\beta = 1$. Further analysis shows that the correlation length exponent is still $\nu = 1/2$. Using the Lifshitz criterion, we then find that the upper critical dimension for percolation is $d_{uc} = 6$, which has been confirmed using RG analysis. The physical origin of this effect is that fluctuations continue to be important at higher dimensions for the percolation model. The percolation model is the simplest example of a quenched random system where the fluctuations are due to disorder in a geometric structure.

In general there are several types of fluctuations, including thermal fluctuations, fluctuations due to quenched disorder, quantum fluctuations etc. In terms of their "strength" it is usually true that disorder has the strongest effect, then thermal fluctuations and finally quantum fluctuations, however there are exceptions to this rule. In any case the fact that disorder or random impurities can have a very strong effect leads to the need for very careful characterization of the defect structure of materials for studies of new phenomena.

That's all folks

Assigned problems and sample quiz problems

Sample Quiz Problems

Quiz Problem 1. Describe the physical meaning of the coherence length (ξ) in superconductors. By considering the linearized Ginzburg-Landau equation in zero field find a solution describing the attenuation of superconducting pair density near the surface of a superconductor.

Quiz Problem 2. Using either London's original argument or starting with the Ginzburg-Landau equation, derive the London differential equation describing the penetration of parallel magnetic field into a superconducting surface. Show that it has the solution $B(x) = B_0 e^{-x/\lambda}$, where λ is the London penetration depth.

Quiz Problem 3. What is the mixed phase of a type II superconductor? Give a physical reasoning to explain why the mixed phase of a type II superconductor can have, at sufficienty high external field, a lower free energy than the Meissner state.

Quiz Problem 4. Write down the scaling assumption for the magnetization and show how it leads to the exponent relations, $\beta = \Delta - \gamma$ and $\Delta = \beta\delta$.

Quiz Problem 5. Write down and explain the scaling assumption used in RG analysis of the ferromagnetic Ising phase transition. Show that if the length rescaling b is taken to be $b = \xi$, then the expected scaling behaviors are recovered.

Quiz Problem 6. Explain the concept of the critical droplet or cluster size in the dynamics of first order phase transitions. Illustrate the discussion by finding the critical droplet size for the ferromagnetic Ising model in an applied field, for temperatures $T < T_c$.

Quiz Problem 7. Discuss the difference between conserved order parameter (COP) and non-conserved order parameter (NCOP) dynamics. For a G-L free energy $F(m)$ for an Ising model, where m is the order parameter, write down the expressions for the COP and NCOP relaxation dynamics of the system. Which relaxation dynamics is slower? Why?

Quiz Problem 8. Write down the Langevin equation for the random motion of a particle. Explain the physical reasoning behind the Langevin approach to dynamics, in particular the noise term. Explain what is mean by "Gaussian white noise". Write down the expressions for the average value of the noise and the correlation function of the noise. Show that the Langevin equation leads to exponential relaxation of the velocity, provided there is no external force applied to the particle.

Quiz Problem 9. Describe the physical processes leading to percolation phenomena. Show that for a random graph the order parameter exponent for percolation is $\beta_{mf} = 1$. Given this, and the values $\nu = 1/2, \eta = 0$, show that the Lifshitz argument indicates that for percolation $d_{uc} = 6$.

Assigned problems

Assigned Problem 1. Starting from the Ginzburg-Landau free energy (Eq. (9)) of the notes, derive the Ginzburg-Landau equation (10).

Assigned Problem 2. Starting from Eq. (9) derive Eq. (11) of the notes.

Assigned Problem 3. By doing a variation of the Helmholtz free energy (15) prove Eq. (17).

Assigned Problem 4. Using London theory, find the Gibb's free energy per unit length of a vortex in a superconductor in an external field H.

Assigned Problem 5. Using the Gibb's free energy within London theory, demonstrate that the triangular lattice vortex array has lower energy than the square lattice array, for a fixed applied field, $H > H_{c1}$, which is close to H_{c1}.

Assigned Problem 6. Prove that $\chi \sim \int dVC(r)$ where for a lattice the integral over volume is replaced by a sum over lattice sites.

Assigned Problem 7. Show that the RG equation for the three dimensional nearest neighbor, ferromagnetic Ising model on a simple cubic lattice, within the Migdal Kadanoff scheme is,

$$Tanh(K') = Tanh^2(3K) \tag{123}$$

Find the fixed points of this RG equation. By carrying out a linear expansion near the fixed point, find the correlation length exponent within this approximation. Is it close to the correct value? Sketch the RG flow for this system.

Assigned Problem 8.
(i) In reduced units ($P_r = P/P_c, T_r = T/T_c, v_r = v/v_c$), the van der Waals equation of state becomes,

$$P_r = \frac{8T_r}{3v_r - 1} - \frac{3}{v_r^2} \tag{124}$$

Show that the spinodal lines of the model are given by,

$$P_r = \frac{3}{v_r^2} - \frac{2}{v_r^3} \tag{125}$$

(ii) By using the condition when $E_{barrier} = 0$, find an expression for the spinodal line of the Ising model on a square lattice, within the droplet theory.

Assigned Problem 9.
(i) Prove the result (107) of notes.
(ii) Fill in the details of the derivation of Eq. (113) from (110).