Statistical Physics (PHY831): Part 4: Superconductors at finite temperature. Introduction to dynamics of many component systems.

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Finite temperature

To extend the ground state calculations to finite temperature, we only need to extend the evaluation of $b_{\vec{k}}$ to finite temperatures, from the discussion at the end of the notes for Part 3, we find,

$$b_k = < a_{-\vec{k}\downarrow} a_{\vec{k}\uparrow} > = u_{\vec{k}} v_{\vec{k}}^* (< 1 - \gamma^\dagger_{\vec{k}\uparrow}\gamma_{\vec{k}\uparrow} - \gamma^\dagger_{-\vec{k}\downarrow}\gamma_{-\vec{k}\downarrow} >) = \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}}(1 - 2f(E_{\vec{k}})) \tag{1}$$

where f is the Fermi function,

$$f(E_{\vec{k}}) = \frac{1}{e^{\beta E_{\vec{k}}} + 1} \tag{2}$$

The *gap equation* at finite temperature is then,

$$\Delta_{\vec{l}} = -\sum_{\vec{k}} V_{\vec{k}\vec{l}} \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}} \left(\frac{e^{\beta E_{\vec{k}}} - 1}{e^{\beta E_{\vec{k}}} + 1}\right) = -\sum_{\vec{k}} V_{\vec{k}\vec{l}} \frac{\Delta_{\vec{k}}}{2E_{\vec{k}}}\ Tanh(\frac{1}{2}\beta E_{\vec{k}}) \tag{3}$$

In the case of the weak-coupling s-wave model, this reduces to,

$$\frac{1}{N(\epsilon_F)V} = \int_0^{\hbar\omega_c} dx\ \frac{Tanh(\frac{\beta}{2}(x^2+\Delta^2)^{1/2})}{(x^2+\Delta^2)^{1/2}} \tag{4}$$

To find the gap at arbitrary temperatures this integral must be evaluated numerically, however, the critical temperature canbe found by setting $\Delta = 0$, so that,

$$\frac{1}{N(\epsilon_F)V} = \int_0^{\beta_c\hbar\omega_c/2} dx\ \frac{Tanh(x)}{x} = ln(x)Tanh(x)|_0^{\beta_c\hbar\omega_c/2} - \int_0^\infty ln(x)Sech^2(x)dx \tag{5}$$

In writing this last expression, we have used the fact that the integrand is now convergent at large x. The upper limit of the integral $\beta_c\hbar\omega_c/2 >> 1$ in all known cases of superconductivity, so we take the upper limit to infinity. In evaluating the first term note that the limit $x \to 0$ of $ln(x)Tanh(x)$ is convergent even though $ln(x)$ diverges as $x \to 0$. The integral that remains is a tabulated definite integral and its value is $ln(4e^\gamma/\pi)$. In the first term we also set $Tanh(\beta\hbar\omega_c/2) \to 1$, again due to the fact that $\beta\hbar\omega_c/2$ is large, then

$$\frac{1}{N(\epsilon_F)V} = ln(\beta_c\hbar\omega_c/2)Tanh(\infty) + a_1 \tag{6}$$

where $a_1 = ln(4e^\gamma/\pi) \sim ln(2 * 1.13..)$ (here $\gamma = 0.577..$ is Euler's constant). Eq. (5) then implies that,

$$k_BT_c = 1.13\hbar\omega_c e^{-1/N(\epsilon_F)V}; \quad \text{and} \quad \frac{2\Delta(0)}{k_BT_c} = 3.52 \tag{7}$$

where we used the zero temperature result $\Delta(0) = 2\hbar\omega_c Exp[-1/(N(E_F)V)]$. The ratio $2\Delta(0)/k_BT_c$ has been checked for a variety of low temperature superconductors and it is quite a good approximation for many of them. Exceptions such as Pb and high T_c materials require either extension of BCS theory or perhaps radically new ideas.

To find the behavior of the gap near T_c, carry out a first order Taylor expansion of Eq. (5) using Δ^2 as the small quantity. This leads to,

$$\frac{1}{N(\epsilon_F)V} = \int_0^{\hbar\omega_c} d\epsilon\ \frac{Tanh(\frac{\beta\epsilon}{2})}{\epsilon} + \Delta^2\int_0^{\hbar\omega_c} d\epsilon\ [\frac{\beta}{4}\frac{Sech^2(\frac{\beta\epsilon}{2})}{\epsilon^2} - \frac{1}{2}\frac{Tanh(\frac{\beta\epsilon}{2})}{\epsilon^3}] \tag{8}$$

This expression may be written in the form,

$$\frac{1}{N(\epsilon_F)V} = ln(\frac{1}{2}\beta\hbar\omega_c) + a_1 + \Delta^2\frac{\beta}{8}a_2 \tag{9}$$

where,

$$a_1 = Ln(4e^{\gamma}/\pi) \ ; \quad \text{and} \ \ a_2 = \int_0^{\infty} dx[\frac{Sech^2(x)}{x^2} - \frac{Tanh(x)}{x^3}] = -0.853 \tag{10}$$

After some algebra, we find that,

$$\frac{\Delta(T)}{k_B T_c} \approx 3.06(1-\frac{T}{T_c})^{1/2}; \qquad T \to T_c \tag{11}$$

Another ratio that is compared to experiment is the contribution of the superconducting transition to the specific heat,

$$\frac{\Delta C}{C_n} = \frac{(C_s - C_n)}{C_n}|_{T_c} = \frac{N(\epsilon_F)}{C_n}(\frac{-d\Delta^2}{dT})|_{T_c} = \frac{(1.74)^2(1.764)^2}{2\pi^2/3} = 1.43 \tag{12}$$

This is a significant additional specific heat, however it corresponds to $\alpha = 0$ as the specific heat does not diverge at T_c.

I. FLUX QUANTIZATION AND VORTEX STATES IN SUPERFLUIDS AND SUPERCONDUCTORS

A. Introduction and a story of two lengths, ξ and λ

Understanding of vortex states requires understanding of two key lengths in superconductors and charged superfluids, the healing length (ξ) and the penetration depth (λ). In neutral superfluids we need to understand the healing length, but there is no analog of the penetration depth. The healing length is the length scale over which superconductivity or superfluidity is suppressed at a surface. Within a BCS picture it is proportional to the size of a Cooper pair. The penetration depth is the depth to which magnetic field penetrates from a normal region into a superconductor. As noted in BCS theory, s-wave superconductors have a singlet spin state that is unfavorable in the presence of a magnetic field, so to maintain the superconducting state, singlet pairing superconductors set up circulating supercurrents to screen out magnetic fields. When magnetic fields are perfectly excluded we are in the perfect diamagnet or Meissner state. In type I superconductors the Meissner state is destroyed at a critical field H_c. However in some superconductors, called type II superconductors, quantized vortices enter the system and form a "mixed phase" consisting of a vortex lattice embedded in a superconducting matrix. In these materials the onset of the mixed state occurs at H_{c1} and the transition to the normal state occurs at H_{c2}. Once we understand the two lengths λ and ξ along with the idea of quantized vortices, we will develop an understanding of the mixed state and estimate the critical fields.

London theory was developed by Fritz London in 1935 to describe the Meissner effect. This theory leads to the introduction of the penetration depth (λ) to describe the extent of magnetic field penetration into superconductors. In type II superconductors λ also describes the extent of flux penetration near vortices. Prior to his studies of superconductivity, Landau had developed a simple mean field theory to describe phase transitions for the Ising and liquid-gas transition (see Part 3 of the course). Ginzburg added a term to describe fluctuations which also enables description of inhomogenious systems. Ginzburg-Landau (GL) (1950) theory is a field theory and provides a systematic phenomenological approach to many body systems. GL theory introduces the healing length or coherence length ξ to decribe the typical length scale of variations in the superfluid density, which applies in particular to supression of superfluidity near surfaces. A theory similar to GL was developed by Gross and Pitaevskii to describe superfluids such as Helium 4 (1961) and we can simply transcribe the healing length result to that case. We first describe the GL theory without a magnetic field, then introduce London theory to describe magnetic field effects and finally introduce magnetic field effects into GL theory and apply it to calculate the upper critical field.

Before proceeding it is important to note that the analysis of London theory and GL theory below uses q for charge, m for mass and n_s for the number density of superconducting electrons. In all superconductors found so far $q = 2e$ is the charge of the fundamental Bosons (Cooper pairs), $n_c = n_s/2$ is the number density of cooper pairs and m is the effective mass of Cooper pairs. In some materials m can be significantly different than $2m_e$ due to band structure effects.

The mixed phase of superfluids and superconductors has received a great deal of attention experimentally. In the case of superfluids, the superfluid is rotated and the rotation leads to the formation of a quantized vortex lattice in the superfluid. It is difficult to see these vortices in superfluid Helium as the healing length in that case is small (about 1 Angstrom). More recently the development of methods to Bose condense cold atomic gases has enabled study of

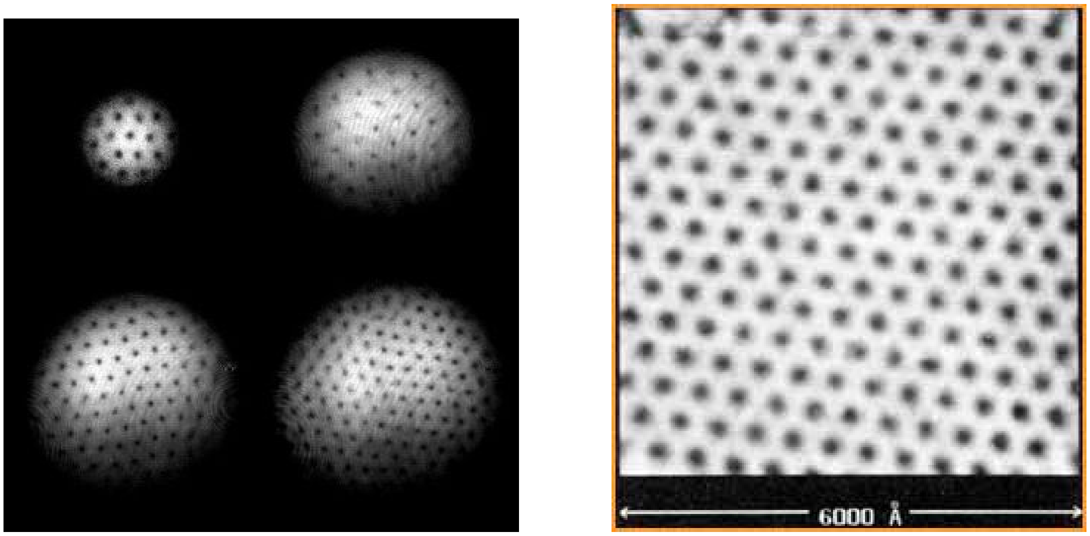

FIG. 1. Left: Vortex lattices in a rotating BEC condensate of cold trapped Na atoms (Ketterle group 2001). Images are for traps with different numbers of atoms and different degrees of external rotation. The triangular lattice structure is evident. Right: Scanning tunnelling microscopy (STM) image of vortices in $NbSe_2$ film at $B = 1$ Tesla and $T = 1.8K$ clearly showing the triangular structure of the vortex lattice (Hess, PRL 1989).

quantized circulated in more detail. In these gases, the healing length can be tuned using density, temperature and trap parameters. In 2001 Ketterle's group was able to clearly image quantized circulated in a trapped Na cold atomic gas (see Figure 1), where the healing length is $0.5\mu m$. Imaging of vortices in superconductors is now quite routine as the healing length (or coherence length) can be quite large (over 100 Angstrom), and the surface of a film can be decorated with small magnetic particles (Bitter pattern) to further define the vortex locations. More recently, scanning tunnelling microscopy provide very clear images of vortex arrays, as illustrated in Figure 1.

B. The healing length ξ

Landau theory is a phenomenological mean field theory to describe behavior near a phase transition. In the case of a superconductor, where the superconducting electrons are described by a "macroscopic" wavefunction, $\psi(\vec{r})$, the Landau free energy is,

$$f_L = f_s(T) - f_n(T) = a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 \tag{13}$$

For $T < T_c$, $a(T) < 0$ and $b(T) > 0$. If we assume that the superconductor or superfluid is uniform, minimizing f_L with respect to ψ yields the physical solution,

$$|\psi_\infty|^2 = \frac{-a}{b}\ ; \quad f_L(|\psi_\infty|) = \frac{-a^2}{2b}, \tag{14}$$

The free energy of a weak coupling isotropic BCS systems can be reduced to,

$$f_{BCS} = f_s - f_n = -N(\epsilon_F)\Delta(T)^2[\frac{1}{2} + ln(\frac{\Delta(0)}{\Delta(T)})] + \frac{\pi^2}{3}N(\epsilon_F)(k_BT)^2 - 4N(\epsilon_F)k_BT\int_0^{\hbar\omega_c} ln(1 + e^{-\beta E})d\epsilon \tag{15}$$

which reduces to $N(E_f)\Delta^2(0)/2$ at zero temperature, so that we can make the connection to the Landau theory through $a^2(0)/2b = N(E_f)\Delta^2(0)/2$. Expanding for small Δ, it can be shown that near the critical point, $f_{BCS} \propto |T - T_c|^2$. We then find that near the critical temperature $a^2(t)/2b = |T - T_c|^2$, so that $a(T) \propto (T - T_c)$ as proposed by Landau. This is also consistent with mean field theory where the specific heat exponent is $\alpha = 0$

In order to add fluctuations (local variations in the wavefunction) to this model, Ginzburg suggested adding a term proportional to $|\nabla\psi(\vec{r})|^2$. There are many ways to motivate this term. Firstly it is the kinetic energy term in quantum mechanics. Secondly it is the lowest order fluctuation term allowed by the symmetry of the order parameter. Thirdly a term like this can be derived directly from the nearest neighbour exchange model of magnetism. The prefactor of this term is often called the "stiffness" as it controls the ability of the material to fluctuate. Adding this term to the free energy (13), we have the Ginzburg-Landau theory in zero field,

$$f_{GL} = f_s(T) - f_n(T) = \int [\frac{\hbar^2}{2m}|\nabla\psi(\vec{r})|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4]dV \tag{16}$$

This has the same form as the mean field form of the Gross-Pitaevskii theory for an interacting Bose gas, as used for example in the study of BEC condensates in cold atom traps. To treat this case, an external potential is included by the replacement $a(T) \to a(T) + V_{ext}$, where V_{ext} is the potential of the atom trap. Minimizing f_{GL} with respect to $\psi^*(\vec{r})$, yields,

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{r}) + a(T)\psi(\vec{r}) + b(T)\psi(\vec{r})|\psi(\vec{r})|^2 = 0. \tag{17}$$

To illustrate the origin of the healing length in either superconductors or superfluids, define $s(x) = \psi(\vec{r})/\psi_\infty$ and consider equation (17) in one dimension, then we have (using $|\psi_\infty|^2 = -a/b = |a|/b$), and considering only the amplitude we have,

$$\xi^2\frac{d^2s(x)}{dx^2} + s(x) - s(x)^3 = 0 \quad \text{where} \quad \xi = (\frac{\hbar^2}{2m|a(T)|})^{1/2} \tag{18}$$

It is clear from this equation that ξ is a length over which the superconducting order parameter fluctuates. It is proportional to the correlation length so we find the correlation length exponent is $\nu = 1/2$, which is the mean field value.

With this theory we can also study the typical length over which superconducting order decays near an air or insulating interface. An interesting solution which shows this explicitly is an interface consisting of an insulator on one side and the other side a superconductor. We consider the interface to be planar, at the origin, and its normal to be in the $\hat{x}$ direction. The boundary conditions that we need are that $s(x \to \infty) = 1$, and $s(x \to -\infty) = 0$. This equation has a rather complicated exact solution, however, the behavior of interest can be found by considering a solution $s = 1 - g$ where g is small. A first order expansion in g of Eq. (18) gives,

$$-\xi^2\frac{d^2g}{d_x^2} = -2g(x); \quad \textit{so that} \quad g(x) \approx e^{\pm\sqrt{2}x/\xi} \tag{19}$$

showing that the order parameter varies on length scales of order ξ as expected.

C. Quantized circulation and flux quantization

Vortices are "topological defects" that are observed in superfluids and superconductors and may be generated in a variety of ways, including thermal excitation. The most controlled way to induce vortices and vortex states in superfluids such as Helium 4 and trapped Bose condensed gases such as Na and Rb gases is to rotate the sample. In superconductors, vortices are generated by magnetic fields beyond a lower critical field, H_{c1}. We start by discussing vortex states in s-wave superconductors and superfluids, though very interesting new effects are expected in p-wave superconductors.

Quantization of circulation and flux are due to the fact that mass or charge current in quantum mechanics can be produced by a gradient in the phase. This is obviously not a classical effect and leads to surprising results. We shall consider two cases: quantized circulation in superfluids and quantized flux in s-wave superconductors.

1. Quantized circulation in superfluids

First we note that circulation, κ is a measure of the vorticity of a fluid and is defined to be,

$$\kappa = \oint \vec{v} \cdot d\vec{l} \tag{20}$$

Below we show that in coherent quantum states κ is quantized.

As a reminder, we first review the origin of the expression for current in quantum mechanics. We consider only the condensed phase described by a wave function,

$$\psi(\vec{r}) = |\psi(\vec{r})|e^{iS(\vec{r})}; \quad where \quad |\psi(\vec{r})|^2 = \psi^*\psi = n_s \tag{21}$$

where n_s is the number density of particles in the superfluid or superconducting state (e.g. superfluid Helium 4 atoms or Cooper pairs). The number density is conserved, so the continuity equation requires that,

$$\frac{\partial(\psi^*\psi)}{\partial t} = -\nabla \cdot \vec{J}_p, \qquad \vec{J}_s = m\vec{J}_p = \rho_s \vec{v}_s \tag{22}$$

where $\vec{J}_s$ is the superconducting mass current, $\vec{v}_s$ is the velocity of the superfluid particles and $\rho_s = mn_s$, where m is the mass of the superfluid or superconducting particles. We write the time dependent Schrodinger equation in the form,

$$i\hbar\frac{\partial\psi}{\partial t} = (-\frac{\hbar^2}{2m}\nabla^2 + V_{NL})\psi \tag{23}$$

where $V_{NL} = a + V_{ext} + b|\psi|^2$ for the Ginzburg-Landau or Gross-Pitaevskii cases discussed above. Using Schrödinger's equation we find,

$$\frac{\partial(\psi^*\psi)}{\partial t} = \frac{\partial\psi^*}{\partial t}\psi + \psi^*\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*). \tag{24}$$

If we define the probability current to be,

$$\vec{J}_P = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*), \tag{25}$$

then the continuity equation (22) is satisfied as it must be. The definition of current in quantum mechanics is thus required by conservation of probability. Now we consider the case where the magnitude of the wavefunction $|\psi|$ is constant so that the mass current deduced from Eq. (25) reduces to,

$$\vec{J}_s = m\vec{J}_P = \hbar|\psi|^2\nabla S = \rho_s\vec{v}_s \quad so\ that \quad v_s = \frac{\hbar}{m}\nabla S \tag{26}$$

where the superfluid density, $\rho_s = m|\psi|^2$.

The demonstration of quantization of circulation is deduced by writing the circulation of the superflow,

$$\kappa_s = \oint \vec{v}_s \cdot d\vec{l} = \frac{\hbar}{m}\oint \nabla S \cdot d\vec{l} = \frac{\hbar}{m}2\pi n = \frac{h}{m}n. \tag{27}$$

where the quantum of circulation is h/m and n is an integer. The key property of quantum systems that leads to this quantization is the fact that the phase change on going around a closed loop must be $2\pi n$, to ensure that the wavefunction is single valued.

2. *Flux quantization in superconductors*

Flux quantization in superconductors or charged superfluids occurs when a magnetic field is applied to the system. To treat this case, we need to generalize the definition of the current to include the vector potential. This is carried out by the substitution,

$$-i\hbar\nabla \to -i\hbar\nabla - qA \tag{28}$$

This substitution comes from classical physics where the canonical momentum for a system in a magnetic field is given by $p \to p + qA$. The origin of this substituion is the definition of the cannonical momentum in classical physics,

$$\vec{p}_{can} = \frac{\partial L}{\partial \vec{v}} \tag{29}$$

where L is the Lagrangian and $\vec{v}$ is the velocity. For a non-relativistic particle in a electro-magnetic field, the Lagrangian is

$$L = \frac{1}{2}mv^2 - qV - q\vec{v}\cdot\vec{A} \tag{30}$$

where V is the scalar potential and $\vec{A}$ is the vector potential. We then find,

$$\vec{p}_{can} = m\vec{v} + q\vec{A} \tag{31}$$

The corresponding Hamilonian is

$$H = \vec{v}\cdot\vec{p}_{can} - L = \frac{1}{2m}(\vec{p}_{can} - q\vec{A})^2 + qV \tag{32}$$

The quantum mechanical procedure is to make the replacement $\vec{p}_{can} \to -i\hbar\nabla$ so the Hamilonian for the quantum case is,

$$H = \frac{1}{2m}(-i\hbar\nabla - q\vec{A})^2 + qV \tag{33}$$

Carrying through the analysis of the current with this Hamiltonian in the Schrodinger equation we find that the current is given by,

$$J_P = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q}{m}|\psi|^2 A \tag{34}$$

Now we again consider the special case where the current is driven only by variations in the phase, so that $|\psi|$ is a constant, we then find that,

$$\vec{J}_q = q\vec{J}_P = \hbar\frac{q}{m}|\psi|^2\nabla S - \frac{q^2}{m}|\psi|^2\vec{A} = q|\psi|^2 v_s \quad so\ that \vec{v}_s = \frac{\hbar}{m}\nabla S - \frac{q}{m}\vec{A} \tag{35}$$

As we demonstrate in the next subsection, superconducting screening currents near a vortex core in a superconductor decrease exponentially at long distances whereas in the superfluid Helium and trapped gas cases the velocity of the superfluid screening currents near a vortex core decay as $1/r$ at long distances. For this reason we can choose a contour at large distances around a vortex in a superconductor where the integral of the superconducting current is zero, so that,

$$\oint \vec{J}_q \cdot d\vec{l} = 0 = \frac{\hbar q}{m}|\psi|^2 2\pi n - \frac{q^2}{m}|\psi|^2 \oint \vec{A}\cdot d\vec{l}, \tag{36}$$

and hence,

$$\oint \vec{A}\cdot d\vec{l} = \phi_B = \frac{h}{q}n = \phi_0 n \tag{37}$$

where $\phi_0 = h/q$ is the flux quantum ($q = 2e$ for Cooper pairs) and n is an integer.

D. An isolated vortex in superfluids and superconductors

Vortices in superfluids and superconductors are topological defects that have quantum circulation (superfluids) or a quantum of flux (superconductors). In both cases there is a vortex core that is in the normal state surrounded by circulating superfluid screening currents that have velocity $v_s(r)$ that depend on the distance from the center of the vortex. In the simplest case the vortices are straight and can be treated using cylindrical co-ordinates. The radius of the vortex core is proportional to the healing length ξ, and we would like to find the rate of decay of the superfluid velocity (superfluid) or current and magnetic field (superconductor) as a function of distance from the center of the vortex.

1. *Superfluid vortex*

In a superfluid we can find the dependence of the superfluid velocity on radius easily, using the definition of the circulation,

$$\kappa_s = \frac{h}{m} n = \oint \vec{v}_s \cdot d\vec{l} = 2\pi r v_s(r) \tag{38}$$

so that for a vortex with a quantum of circulation,

$$v_s(r) = \frac{h}{2\pi m r} \tag{39}$$

The energy of the vortex is composed of the energy of the core and the kinetic energy of the circulating superfluid. The dominate term is the kinetic energy of the superfluid and this energy is given by,

$$\epsilon_{kin} = \int_\xi^b \frac{1}{2}\rho_s v_s^2 2\pi r dr = \frac{h^2 \rho_s}{4\pi m^2} ln(b/\xi) \tag{40}$$

where b is the size of the sample. We shall also need the angular momentum of each vortex, and a straightforward calculation gives,

$$\vec{l}_1 = \frac{1}{2}\rho_s \kappa_0 b^2 \tag{41}$$

where $\kappa_0 = h/m$ is the quantum of circulation.

2. *Superconducting vortex*

We combine Maxwell's equation for the current with the quantum mechanical expression (35) to find,

$$\vec{J}_q = \frac{1}{\mu_0} \nabla \wedge \vec{B} = \frac{\hbar q}{m} |\psi|^2 \nabla S - \frac{q^2}{m} |\psi|^2 \vec{A}. \tag{42}$$

Taking a curl and using the fact that curl(grad(S)) = 0 and that $\nabla \wedge \nabla \wedge B) = \nabla(\nabla \cdot B) - \nabla^2 B$ with $\nabla \cdot B = 0$, we find London's equation,

$$\nabla^2 B = \frac{B}{\lambda^2} \tag{43}$$

where λ is the London penetration depth and characterizes the depth to which magnetic field penetrations from a normal region into a superconductor. Here we defined,

$$\lambda = (\frac{m}{n_c q^2 \mu_0})^{1/2} (MKS) \quad \text{or} \quad \lambda = (\frac{mc^2}{4\pi n_c q^2})^{1/2} (CGS). \tag{44}$$

The simplest case is penetration of magnetic field into a superconductor from a flat interface, in which case we find,

$$B = B_0 e^{-x/\lambda}; \qquad J = J_0 e^{-x/\lambda} \tag{45}$$

where the current is found from the field using Maxwell's equation.

To find the magnetic field penetration and screening currents for a vortex, we solve in cylindrical co-ordinates where the radial part of Eq. (43) is,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial B}{\partial r}) - \frac{B}{\lambda^2} = 0 \ \ or \quad -\frac{\phi_0}{\lambda^2}\delta(r), \tag{46}$$

where r lies in the x-y plane. We can impose the boundary condition at the core using a source (delta function) or through the boundary condition $B(0) = B_0$ and then integrate to impose flux quantization. Solving this system we find that the magnetic field penetration from a vortex core is given by the Bessel function,

$$B(r) = B_0 K_0(\frac{r}{\lambda}) \quad \text{in the z direction} \tag{47}$$

The Bessel function $K_0(x)$ has the following behaviors:

$$K_0(x \to 0) \sim -ln(x/\lambda) \ ; \quad K_0(x \to \infty) \sim e^{-x} \tag{48}$$

It is clear that this is unphysical as $x \to 0$, but this is reasonable as this theory is invalid for distances $r < \xi$. It is meant to describe behavior on length scales $r >> \xi$. For many type II superconductors the London approximation is very good because $\lambda >> \xi$. Current circulation is induced by the magnetic field gradient according to the Maxwell equation $\mu_0 J = \nabla \wedge B$, this yields,

$$J(r) = J_0 K_1(\frac{r}{\lambda}) \qquad \text{in the } \theta \text{ direction} \tag{49}$$

This Bessel function is related to $K_0(x)$ via, $K_1(x) = -dK_0(x)/dx$. It has the following limiting behaviors,

$$K_1(x \to 0) \sim \frac{1}{x} \ ; \qquad K_1(x \to \infty) \sim e^{-x} \tag{50}$$

At distances $r < \lambda$ vortices in superconductors look a lot like vortices in superfluid Helium II. However at long distances $r > \lambda$ they are screened and the magnetic field and current decay exponentially.

There is only one parameter remaining in the construction above, and that is the magnetic field B_0. This is set by the requirement that the flux be quantized, that is,

$$\int_0^\infty B_0 K_0(\frac{r}{\lambda}) 2\pi r dr = \phi_0 \tag{51}$$

where $\phi_0 = h/q$ is the flux quantum. This is a tabulated integral (e.g. Mathematica can do it),

$$\int_0^\infty x K_0(x) dx = 1 \tag{52}$$

Solving for B_0 yields

$$B_0 = \frac{\phi_0}{2\pi\lambda^2} \ ; \text{which implies} \quad J_0 = \frac{\phi_0}{2\pi\mu_0\lambda^3} \tag{53}$$

The *Helmholtz free energy*(per unit length) of an isolated vortex is given by,

$$f_1 = \frac{1}{2\mu_0} \int_0^\infty (B^2 + \lambda^2 \mu_0^2 J^2) 2\pi r dr \tag{54}$$

which explicitly shows the contributions of the field (first term) and the current (second term). The second term is of the same form as the energy we used in the superfluid case and is due to the kinetic energy of the supercurrents that circulate around the vortex core. In the large λ/ξ limit, this is dominated by the regime $\xi \leq r \leq \lambda$, so we find an approximate value of the vortex energy by using,

$$\begin{aligned} f_1 &\approx \frac{1}{2\mu_0} (\frac{\phi_0}{2\pi\lambda^2})^2 2\pi \int_\xi^\lambda [r(Ln(\frac{\lambda}{r}))^2 + r(\frac{\lambda}{r})^2] dr \\ &= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} \int_{\xi/\lambda}^1 [x(Ln(x))^2 + \frac{1}{x}] dx \\ &= \frac{\phi_0^2}{4\pi\mu_0\lambda^2} [\frac{x^2}{2}(\frac{1}{2} - Ln(x) + Ln(x)^2) + Ln(x)]|_{\frac{\xi}{\lambda}}^1 \\ &\approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2} Ln(\frac{\lambda}{\xi}) \end{aligned} \tag{55}$$

Notice that the energy cost of forming the vortex is dominated by the kinetic energy of the superconducting electrons (the logarithmic term). The energy cost due to the magnetic field is relatively small. In addition we should add the energy cost of the normal core of the vortex. This energy is

$$\epsilon_{core} \approx \pi\xi^2 \frac{N(\epsilon_F)\Delta^2}{2} = \pi\xi^2 b \frac{a^2}{2b^2} = \pi\xi^2 b \frac{|\psi|^2}{2} = \pi\xi^2 \frac{mb}{\mu_0 q^2 \lambda^2} \tag{56}$$

where the last expression is found using Eq. (44). For large ratios of λ/ξ this core energy is also relatively small compared to the kinetic energy of Eq. (55). However as we shall see later the core energy is very important in pinning of vortices, which is essential to applications to superconducting magnets.

E. Conditions for vortex states

Figure 1 shows pictures of vortex states in superfluids and superconductors and one can imagine many situations where quantized vortices might occur as any coherent quantum system must obey the phase quantization condition used in their derivation. However vortices only enter a system when the conditions are tuned to make vortex states the lowest free energy state. In the superfluid example, the rotation of the superfluid must be high enough to produce the vortex state and in the superconductor example, the magnetic field must be high enough to stabilize the vortex state. We now discuss more precise conditions for the onset of vortex states.

1. The critical rotation speed for vortex states in superfluids

When a quantum fluid is rotated at angular frequency Ω it tries to distribute its circulation uniformly. A rigid body rotates with constant angular speed so that,

$$\nabla \wedge \vec{v} = 2\vec{\Omega}. \tag{57}$$

A superfluid distributes quantized vortices that have circulation κ_1 uniformly throughout the superfluid so that the circulation is uniform. The number of vortices is then,

$$n_v = 2\Omega/\kappa_0, \quad where \quad \kappa_0 = h/m. \tag{58}$$

For this case, the angular momentum per particle is $N_v\hbar/2$, where N_v is the number of vortices in the system. To find the critical angular frequency for the onset of vortex states we need to minimize the correct free energy. In this problem, the applied angular frequency is a reservoir of angular momentum and the correct free energy to minimize is $F - \vec{L}\cdot\vec{\omega}$, where $\vec{L}$ is the angular momentum of the superfluid and $\vec{\Omega}$ is the angular velocity of the container. The critical condition is then,

$$f_1 - \vec{l}_1 \cdot \vec{\omega} = 0 \tag{59}$$

so that (using equations (40) and (41)),

$$\Omega_c = \epsilon_1/l_1 = \frac{k_0}{2\pi b^2} ln(b/\xi) \tag{60}$$

which shows that in large systems vortices are always present, while in small systems it is harder to generate them. This is also true in classical turbulence.

2. The lower critical field H_{c1} of superconductors

The difference in Gibb's free energy between a superconductor containing no vortices and a superconductor containing one vortex, for a field parallel to a thick slab in the large λ/ξ limit, is given by,

$$\delta g_{GL} = f_1 - \int_0^\infty 2\pi r dr B\cdot H = f_1 - \phi_0 H \tag{61}$$

The lower critical field is determined by when $\delta g_{GL} = 0$. For fields higher than this, it is favorable for flux to enter the superconductor while for fields lower than this, the Meissner state is favored. In the limit of large λ/ξ, we can use the vortex energy given in Eq. (55) so that,

$$H_{c1} = \frac{\epsilon_1}{\phi_0} \approx \frac{\phi_0}{4\pi\mu_0\lambda^2} Ln(\frac{\lambda}{\xi}) \tag{62}$$

Just above H_{c1} there is a rapid influx of vortices as the interaction between vortices is relatively weak until their separation is less than λ. The vortices also pack efficiently to maximize the decrease in magnetization, which favors the triangular stacking of vortices (this is typical for central force systems).

F. Adding a magnetic field to GL theory

Many applications of GL theory are to the analysis of the effects of an applied magnetic field. In the presence of a magnetic field, the Helmholtz free energy above must be extended in two ways. Firstly, the momentum operator $p = -i\hbar\nabla$ is replaced by $p \rightarrow -i\hbar\nabla - qA$, where A is the vector potential associated with the magnetic field B, and $-q$ is the charge of the cooper pair. We use the following definitions $m = 2m^*$ (where m^* is the effective mass of the electron), $q = 2e$, $|\psi|^2 = n_s$, where n_s is the density of superconducting electrons. Secondly, the field energy has to be added to the free energy. These modifications lead to the expression,

$$f_{GL} = \int dV[\frac{1}{2m}|(-i\hbar\nabla - qA)\psi(\vec{r}))|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} - \frac{\mu_0 H^2}{2}] \tag{63}$$

The term $B^2/2\mu_0$ is the magnetic field energy inside the superconductor while $\mu_0 H^2/2$ is the field energy in the normal state.

However it is wrong to use the Helmholtz energy in calculations where the external field controls the electrodynamics. This is because we must take into account the amount of energy required to set up the applied field as well. The free energy we need to use is the Gibb's free energy $g = f - \mu_0 H \cdot M$. Where M is the magnetisation. If we take the normal state magnetisation to be zero, and use, $B = \mu_0(H + M)$, we find that the Gibb's free energy is,

$$g_{GL} = \int dV[|\frac{1}{2m}(-i\hbar\nabla - qA)\psi(\vec{r}))|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B \cdot H]. \tag{64}$$

Note that in many superconducting texts and papers M is taken to have the units of B. I am using the conventional magnetostatics definition. Within mean-field theory, we assume that the order parameter takes on a value which optimizes the above free energy. In this expression we can optimize the wavefunction and the field. We thus do a variation with respect to the wavefunction (or ψ^*) to produce the GL equation. In addition if we carry out a variation with respect to the vector potential A which we find the quantum mechanical expression for the current. Using the Euler-Lagrange equations to do the variation with respect to $\psi*$ yields,

$$a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - qA)^2\psi = 0 \tag{65}$$

A variation with respect to the vector potential yields (and using $\mu_0 j = \nabla \wedge B$, and the gauge $\nabla \cdot A = 0$), we find,

$$j_s = \frac{-iq\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m}A|\psi|^2$$

$$= \frac{q}{m}|\psi|^2(\hbar\nabla S - qA) = q|\psi|^2 v_s \tag{66}$$

The electromagnetic properties of superconductors are determined by Eqs. (9-11)) in combination with Maxwell's equations. A good starting point to understand the electrodynamics of superconductors is London theory, that sets $|\psi| = constant = |\psi_\infty|$, that is, the amplitude of the order parameter is assumed to be a constant.

G. The thermodynamic critical field, H_{cb}

The thermodynamic or bulk critical field H_{cb} is the field at thich the field energy is the same as the condensation free energy, f_{cond}. We calculated the condensation free energy within Landau theory (see Eq. (2)), and the BCS expression for this energy is given in Eq. (3). From these expressions, we find that $f_{cond} \propto |T - T_c|^2$ near T_c for both BCS theory and Landau theory, as expected for mean field calculations where $\alpha = 0$. The field energy is $\mu_0 H^2/2$, which is the field energy required to expel flux from the interior of a superconductor, as occurs in the Meissner phase. This calculation assumes a field applied parallel to a slab of superconductor with thickness $t >> \lambda$. In that case the thermodynamic critical field is given by,

$$\frac{\mu_0 H_{cb}^2}{2} = f_{cond} \sim |T - T_c|^2. \tag{67}$$

In this analysis we have considered only two states, the normal state with uniform flux and the Meissner phase of a superconductor. It turns out that for type I superconductors this is correct and there is a first order transition from the Meissner phase to the normal phase. The phase diagram then consists of just those two phases. If the applied field/sample geometry is not a parallel field applied to a slab, demagnetization effects can lead to complex flux penetration patterns. This regime that is sample geometry dependent is called the intermediate phase and has also generated many studies.

Type II superconductors do not make a direct transition from the Meissner phase to the normal phase, and instead make the transition through an intermediate phase called the mixed phase. To estimate the value of κ that separates Type I superconductors from type II superconductors, we consider a lamellar mixed phase, as originally considered by Landau. We consider whether it is energetically favorable to add interfaces into the material at H_{cb}. The Gibb's free energy to add an interface is approximately,

$$G_{interface} = f_{cond} L^2 \xi - \frac{1}{2}\mu_0 H^2 L^2 \lambda; \quad \text{so that,} \quad \frac{G_{interface}}{L^2} = -\frac{1}{2}\mu_0 H_{cb}^2 (\lambda - \xi) \tag{68}$$

Clearly the superconducting phase at H_{cb} is unstable to the formation of interfaces provided $\lambda > \xi$, or $\kappa > 1$. Abrikosov showed that the exact critical value, $\kappa_c = 1/2^{1/2}$, that is extracted by the analysis of vortex lattices which is the correct morphology of the mixed phase.

In type I superconductors where $\kappa < 1/2^{1/2}$ there is one critical field H_{cb} at which the flux suddenly penetrates the sample (ignoring demagnetisation effects) while in type II superconductors there are two critical fields: H_{c1} when flux quanta first penetrate a sample and H_{c2} when the applied field finally destroys superconductivity (at H_{c2} the fluxons pack so densely that their cores overlap). We have already determined H_{cb} and below we shall find the two critical fields H_{c1}, H_{c2} using GL theory in the limit of large κ (extreme type II). Before doing that we need to develop a more complete understanding of an isolated vortex.

The discussion below ignores the effect of random pinning on the flux states, which is only valid in the cleanest materials. The fields we find are the "reversible" critical fields. In most type II materials pinning is important and there is considerable hysteresis in the magnetisation. In these cases, one can define a variety of irreversibility lines. All commercial magnets and proposed transmission line applications of superconductors require good flux pinning. Vortices are also called flux lines, and they experience a Lorentz force when a DC current flows through a superconductor. If there is no flux pinning, the vortices move leading to an induced emf, through Faraday's/Lenz law.

H. The upper critical field H_{c2}

At the upper critical field we can make several approximations which make the GL equations much simpler. Firstly the order parameter ψ is small so we can ignore the non-linear term. Secondly, we can assume that the vector potential inside the superconductor is nearly at the value specified by the external field, e.g. $A = (0, \mu_0 H x, 0)$ we shall use this choice of potential for convenience though other choices which satisfy $\nabla \cdot A = 0$ are equally valid due to gauge invariance. With these approximations and assumptions, the GL equation reduces to,

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2m}(-i\hbar\frac{\partial}{\partial y} - q\mu_0 H x)^2 \psi - \frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial z^2} = -a\psi \tag{69}$$

This is the Schrödinger equation for a particle in a magnetic field in the z-direction. The energy eigenvalues are known to be,

$$|a| = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar^2 k_z^2}{2m} \tag{70}$$

where ω_c is the cyclotron frequency,

$$\omega_c = \frac{q\mu_0 H}{m} \tag{71}$$

However we must interpret Eq. (70) in an unusual way. We know $|a|$ and we must find the *largest* applied field H that corresponds to it. The largest field occurs when $k_z = 0$ and $n = 0$, so we have the simple result (using (71) in (70), with $k_z = n = 0$),

$$H_{c2} = \frac{2m|a|}{\hbar\mu_0 q} = \frac{\phi_0}{2\pi\mu_0\xi^2} \tag{72}$$

where we have used $\xi^2 = \hbar^2/(2m|a|)$ to find the last expression in Eq. (72). Since $|a| \sim |T - T_c|$, the critical field, H_{c2} approaches zero linearly near the critical temperature. Similarly, the lower critical field also approaches zero linearly near the critical temperature. Finally we have the interesting result (taking the ratio of Eqs. (72) and (70)),

$$\frac{H_{c2}}{H_{c1}} = 2(\frac{\lambda}{\xi})^2 \frac{1}{Ln(\lambda/\xi)} \tag{73}$$

This expression demonstrates that even for moderate values of λ/ξ, the two critical fields H_{c2} and H_{c1} are well separated.

II. STATISTICAL DYNAMICS

Many of the concepts and methods that we introduced in the study of equilibrium properties of statistical systems can be extended without much difficulty to the dynamics case. For example in many cases if we perturb a system away from equilibrium, the driving force toward equilibrium is given by the deviation of the chemical potential from its equilibrium value, that is if μ_0 is the chemical potential at equilibrium, we can prepare a system at a different density, pressure or particle number so that the system is in a non-equilibrium state with chemical potential difference, $\mu - \mu_0$ ¿0. We can then study the relaxation toward equilibrium. If the system is reasonably close to equilibrium, then we can use the expressions for a free energy surface that we have developed for the equilibrium problems to write,

$$\mu = \frac{\delta free\ energy}{\delta X} \tag{74}$$

where we have to choose the Hemlholtz or appropriate Gibb's free energy depending on the system. X is the variable that we are perturbing from its equilibrium value. For example for relaxation of the magnetization of an Ising model with $T > T_c$, we cound use the Landau Free energy so that,

$$\mu = a(T)m + b(T)m^3 \tag{75}$$

Relaxation of the magnetization from a small finite value m_0 at $T > T_c$ with $h = 0$ is then given by,

$$\frac{\partial m}{\partial t} = -\Gamma\mu = -\Gamma(a(T)m + b(T)m^3) \tag{76}$$

When m is close to zero, this can be solved to find,

$$m(T) = m_0 e^{-t/\tau_I} \quad \tau_I = 1/(\Gamma a(T)) \tag{77}$$

This shows that relaxation is exponential with a relaxation time that depends on proximity to T_c. On approach to T_c the relaxation time diverges, which is the phenomenon of "critical slowing down".

However to describe the relaxation dynamics in general we have to consider different types of dynamics and there are many possibilities. Below we shall discuss two important types of relaxation, namely non-conserved and conserved order parameter dynamics. Sometimes these two classes of problem are called Allen-Cahn and Cahn-Hilliard respectively, and in statistical physics they are also referred to as Glauber and Kawasaki dynamics. The distinction between conserved and non-conserved dynamics is understood easily when considering the relaxation of a surface and we shall discuss that case below. It is generally true that conserved dynamics leads to slower relaxation than non-conserved dynamics. After discussing simple cases of relaxation after a weak perturbation we shall look at dynamics of formation of a new equilibrium phase. For example if we cool water vapor below the condensation temperature water droplets form. The dynamics of formation of the liquid phase occurs by nucleation and growth, a process that is important in all branches of physics and technology. The growth dynamics often occurs by motion of interfaces and depends on whether the dynamics follows a conserved or a non-conserved process. The former process may lead to the famous Lifshitz-Slyosov cluster dynamics that describes many coarsening phenomena.

We talked about several dynamical methods in Part 1 of the course including the Liouville equation, the Master equation and molecular dynamics simulations. These methods form the fundamental basis of statistical dynamics. Another approach we mentioned briefly is kinetic theory which at its most basic is like a mean field theory for particle transport. We shall start our discussion of statistical dynamics with a discussion of the Langevin approach to kinetic theory and show that it leads to the prediction of exponential relaxation and also a fluctuation-dissipation relation. We shall then show how the relaxation dynamics driven by chemical potential also leads to exponential relaxation. Dynamics driven by chemical potential differences will then be discussed to show the difference between conserved and non-conserved dynamics. Combination of dynamics driven by chemical potential differences and random forces used in Langevin theory leads to very general equations for dynamics that will be discussed briefly.

A. Langevin equation and linear response

The Langevin approach reduces the N-particle phase space dynamics to a lower dimensional dynamics. If we consider $M << N$ of the particles in an N dimensional system, we can consider each of the M particles to be essentially independent. Then we are interested in the dynamics of one particle in a reservoir of other particles that are assumed to be in a state of "molecular chaos". In the simplest case, Langevin theory considers the effect of molecular chaos to be a random force, but with the added feature that the perturbations due to the random forces are damped. We then have the basic Langevin equation for one particle in an N-particle system,

$$m\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{B} + \eta'(t) \tag{78}$$

where B is the damping coefficient and $\eta'(t)$ is the random force. The damping coefficient is equal This force is often assumed to be "Gaussian white noise", which means that there are no correlations in the noise, either in time or space. Colored noise introduces correlations in the noise. Note that for conserved dynamics problems the noise must include the effects of the conservation laws. If there is no net direction or flow in the particle dynamics of the system then $< \eta'(t) >= 0$. The Langevin equation provides a coupling between equilibrium properties (the fluctuations in the random force) and non-equilibrium properties (the damping B).

Langevin theory is very nice in that it gives a very direct way to find the relation between the diffusion constant and the damping coefficient. This is demonstrated by solving for the mean square distance that a particle moves, according to the Langevin equation and to calculate a similar quantity from the diffusion equation. The diffusion equation states that,

$$\frac{\partial n}{\partial t} = D\nabla^2 n \quad \text{so that} \quad n = \frac{1}{(4\pi Dt)^{d/2}} e^{-|\vec{r}-\vec{r}_0|^2/4Dt} \tag{79}$$

where n is the single particle density and is normalized to one, d is the spatial dimension, and D is the diffusion constant. From this expression the we find

$$< |\vec{r} - \vec{r}_0|^2 >= 2dDt. \tag{80}$$

Now lets analyse the stochastic Langevin equation. If we take a time average of Eq. (106), we find the simple result,

$$< v(t) >= v(0)e^{-t/\tau} \quad \text{where} \quad \tau = mB \tag{81}$$

is the relaxation time. That is, the relaxation of the velocity to equilibrium is exponential. If there is no net flow, as assumed here, at long times the average velocity is zero in agreement with the MB distribution. With this definition the Langevin equation becomes,

$$\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{\tau} + \eta(t) \quad \text{where} \quad \eta(t) = \eta'(t)/m. \tag{82}$$

We find the mean square distance travelled by the particle using the Langevin equation, by taking a dot product of $\vec{r}$ with the Langevin equation and the averaging over the noise. To carry this out, we use the results,

$$\vec{r}\cdot\vec{v} = \frac{1}{2}\frac{d(\vec{r}\cdot\vec{r})}{dt}; \qquad \vec{r}\cdot\frac{d\vec{v}}{dt} = \frac{1}{2}\frac{d^2(\vec{r}\cdot\vec{r})}{dt^2} - v^2; \qquad < \vec{r}\cdot\eta(t) >= 0.0 \tag{83}$$

to find that,

$$\frac{d^2 < r^2 >}{dt^2} + \frac{1}{\tau}\frac{d < r^2 >}{dt} = 2 < v^2 >= \frac{2d_s k_B T}{m} \tag{84}$$

where d_s is the spatial dimension (e.g. 3) and the last expression comes from using the equipartition theorem for a monatomic particle system. Solving this equation yields,

$$< r^2 >= \frac{2d_s k_B T}{m}\tau^2[\frac{t}{\tau} - (1 - e^{-t/\tau})] \quad \text{so as } t \to \infty \qquad < r^2 >= 2d_s B k_B T t \tag{85}$$

Comparing with the diffusion equation result yields $B = D/k_BT$, which is the Einstein relation for brownian motion. It is one example of a fluctuation-dissipation result where the close to equilibrium response (the damping) is related to the dynamical fluctuations at equilibrium (the diffusion). The damping coefficient B can be measured as the "mobility" defined as the ratio of the terminal velocity over the driving force.

1. Non-conserved and conserved dynamics driven by domain wall free energy

A variety of different dynamical processes can lead to the same equilibrium state, however the rate at which equilibrium is approached depends strongly on the type of dynamics. There are two basic classes of dynamics of phase transitions though there are many subclasses. Ginzburg-Landau mean field theory may be extended to treat both conserved and non-conserved dynamics of the order parameter characterizing a system. In the case of a ferromagnet the dynamical GL equations describe relaxation of the magnetization, while in a superconductor they describe the relaxation of the order parameter magnitude and phase. The simplest example to consider is relaxation of a surface in one dimension where the height of the surface is characterized by $h(x)$. This can describe for example the interface between an up spin domain and down spin domain in a ferromagnet. We consider first a continuum model to describe the relaxation of a perturbation of a domain wall in two dimensional Ising ferromagnet. We assume that energy of the domain wall depends on its length times its energy per unit length, so that,

$$E_{DW} = \sigma \int dx \sqrt{1+(\frac{dh}{dx})^2} \approx constant + \frac{\sigma}{2}\int dx (\frac{dh}{dx})^2 \tag{86}$$

where the last expression is found by making the small angle approximation to expand the square root to leading order. Here σ is the surface tension. We assume that it is isotropic, which is only true close to the critical temperature. For a ferromagnetic Ising model at low temperature $\sigma \approx 2J$. The equilibrium state is found by doing a variation of the domain wall energy using the Euler-Lagrange equations. To study relaxation processes, we consider two cases. The simplest case, called Allen-Cahn theory, is to assume a simple non-conserved relaxation dynamics where,

$$\frac{\partial h}{\partial t} = -\gamma\mu = \gamma \frac{\delta E_{DW}}{\delta h} = \gamma\sigma \frac{\partial^2 h}{\partial x^2} \tag{87}$$

where we introduced the rate γ, and chemical potential μ. This is a course grained approach to dynamics in the same sense that GL theory is a coarse grained approach to equilibrium phase transitions. In fact, later we shall extend the GL model to include dynamics of this type. This is the diffusion equation, so we expect that length and time are related through $x \propto t^{1/2}$. A general scaling approach to partial differential equations illustrates this scaling. We assume the form,

$$h(x,t) = t^{-a} f(x/t^b) = t^{-a} f(s); \qquad \text{where} \qquad s = x/t^b \tag{88}$$

Substituting this expression into the diffusion equation gives,

$$-at^{-a-1}f(s) - bxt^{-a-b-1}f'(s) = -at^{-a-1}f(s) - bt^{-a-1}sf'(s) = \gamma\sigma t^{-a-2b}f''(s) \tag{89}$$

We now choose $b = 1/2$ so that the time dependence drops out of the equation. This proves that all solutions of this form have the relationship $x \sim t^{1/2}$ where $f(s)$ is the scaling function or shape function which satisfies,

$$-af(s) - bsf'(s) = \gamma\sigma f''(s) \tag{90}$$

This type of dynamics is typical of Ising domain walls where the number of up and down spins are not conserved.

Now consider a converved domain wall dynamics which is often called Cahn-Hilliard theory, where the number of particles is conserved so the relaxation occurs by transport of particles along the surface through a surface current (Fick's law,

$$\vec{j} = -k\nabla\mu \to j = -k\frac{\partial \mu}{\partial x} \tag{91}$$

where k is a constant. The continuity equation ensures conservation of particle number, so that,

$$\frac{\partial h}{\partial t} + \nabla\cdot\vec{j} = 0 \tag{92}$$

so that for a one dimensional interface,

$$\frac{\partial h}{\partial t} = -\gamma_1 k \frac{\partial^4 h}{\partial x^4} \tag{93}$$

A similarity analysis of this equation gives,

$$-at^{-a-1}f(s) - bxt^{-a-b-1}f'(s) = -at^{-a-1}f(s) - bt^{-a-1}sf'(s) = -\gamma_1 k t^{-a-4b}f''''(s) \tag{94}$$

We now choose $b = 1/4$ so that the time dependence drops out of the equation. This proves that all solutions of this form have the relationship $x \sim t^{1/4}$ where $f(s)$ is the scaling function or shape function which satisfies,

$$-af(s) - bsf'(s) = -\gamma_1 k f'''' \tag{95}$$

Clearly the dynamics of the convserved case is much slower due to the requirement that transport is only along the surface, instead of through exchange with a reservoir.

B. Time dependent G-L theory

The Langevin concept of adding a noise term to include the effect of fluctuations is often used to consider the effects of fluctuations on the dynamics of the Ginzburg-Landau theory. To do this the COP and NCOP theories are written for the order parameter, and a noise term is added, leading to,

$$\frac{\partial X}{\partial t} = -\Gamma_1 \frac{\delta F}{\delta X} + \eta_N(x,t) \qquad (NCOP); \qquad \frac{\partial X}{\partial t} = \Gamma_2 \nabla^2 \frac{\delta F}{\delta X} + \eta_C(x,t) \qquad (COP); \tag{96}$$

where F is the G-L free energy of the system, which can be the Gibb's, Helmholtz or other free energy appropriate to the system under study. These are quite general equations where the noise must obey the conservation laws of the system. For the case of Gaussian white noise we have,

$$< \eta_N(x,t)\eta_N(x',t') >= 2k_B T \Gamma_1 \delta(x,x')\delta(t,t'); \qquad < \eta_C(x,t)\eta_C(x',t') >= 2k_B T \nabla^2 \Gamma_2 \delta(x,x')\delta(t,t') \tag{97}$$

III. DYNAMICS OF FORMING A NEW PHASE: SPINODAL LINES AND CRITICAL DROPLETS

Above we discussed the relaxation of a surface and the relaxation of magnetization from a perturbed state to the lowest free energy state. However in many situations, a new phase forms from an existing phase due to a change in temperature, density or field. This occurred in the early universe when matter was formed, it happens in clouds to cause rain and it is essential to know how to control it when we want to change the spin orientation of a magnetic storage bit in our computers. We will look at the dynamics of new phase formation using the liquid gas transition and Ising model as examples.

In general, formation of a new phase may occur by nucleation and growth or by spinodal decomposition. In the latter case, there is no free energy barrier to formation of the new phase, whereas in the former a free energy barrier exists so the new phase is formed by nucleation and growth. In both cases, of course, the new phase must be a lower free energy state that the old phase. The boundary between the regime where nucleation and growth occurs and the regime where spinodal decomposition occurs is called the spinodal line and is relatively easily found within mean field theory. In the van der Waals theory, we find the points at which $\partial P/\partial v = 0$, while in the Ising model we find the onset of a metastable state in the Landau equation,

$$a(T)m + b(T)m^3 - h = 0. \tag{98}$$

The condition for the onset of three real roots in these equations is equivalent to the spinodal condition (onset of a metastable state), which in turn is when the discriminant of the cubic is zero. The discriminant of a general cubic,

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0 \tag{99}$$

is

$$\Delta = \beta^2\gamma^2 - 4\alpha\gamma^3 - 4\beta^3\delta - 27\alpha^2\delta^2 - 18\alpha\beta\gamma\delta \tag{100}$$

In the case of the Ising model $\alpha = b, \beta = 0, \gamma = a, \delta = -h$, so the spinodal condition is,

$$h_{spin} = (\frac{4a_0(T_c - T)}{27b})^{3/2}; \quad where \quad a(T) = a_0(T - T_c) \tag{101}$$

The spinodal lines in the van der Waals theory are found by solving,

$$\frac{\partial P}{\partial p} = 0 = -\frac{k_B T}{(v-b)^2} + \frac{2a}{v^3} \tag{102}$$

so,

$$-k_B T v^3 + 2av^2 - 4abv + 2ab^2 = 0 \tag{103}$$

If we quench into an unstable or spinodal regime of any system, formation of a new phase is limited purely by downhill dynamics and the morphologies that are generated are usually quite random. This is called the "spinodal decomposition" regime and is useful if interpenetrating or percolating morphologies are desirable. However in most cases we would like a more orderly growth or formation of a new phase, and in the case of crystal growth extremely high purity and orderly growth. To achieve this goal the degree of "supersaturation" is kept small and often a nucleation center is placed in the system, which in the case of single crystal growth is often a seed single crystal. In the case of nucleation of water droplets from water vapor cloud seeding works in some case. In the case of flipping a domain in a magnetic storage bit, the growth of the new phase is often by domain wall nucleation at the edges of the domain following by domain wall motion through the cluster. Here we will look at one case, the critical droplet size in homogeneous nucleation.

Nucleation of a new phase occurs when the system is in a metastable phase with chemical potential μ_{old} and there is a new phase with chemical potential μ_{new} that is lower in free energy. However to make the transition to the new phase, there must be a dynamical mechanism for the formation of the new phase from the old phase. The dynamical mechanism may be nucleation of the new phase at the boundaries, at heterogeneities in the bulk, it can be from highly ordered seed nuclei, or it can be from thermally activated domains of the new phase which is called homogeneous nucleation. We only consider homogeneous nucleation. Consider a droplet of the new phase in the old phase. The free energy of a spherical droplet of the new phase is,

$$g_{droplet} = (\mu_{new} - \mu_{old})\frac{4\pi}{3}R^3 + \sigma 4\pi R^2 \tag{104}$$

where the first term is the volume term and is favorable while the second term is the surface term and is unfavorable. Here σ is the surface energy cost per unit area of an interface between the new phase and the old phase. Small droplets have positive energy due to the surface term so they are themally activated and then disappear. Large droplets are dominated by the bulk term and once they form they grow. The critical droplet size is the size above which the droplets grow. To find the critical droplet size we minimize the droplet free energy with respect to the radius to find,

$$R_c = \frac{2\sigma}{|\delta\mu|}; \quad and \quad g_{droplet}(R_c) = g_{barrier} = \frac{4\pi}{3}\sigma(\frac{2\sigma}{|\delta\mu|})^2 \tag{105}$$

In the case of a Ising magnetic system where the initial state is the up spin state for temperatures $T << T_c$, and we apply a magnetic field $h < h_{spin}$, the energy of a droplet of size R is approximately,

$$E_{droplet} \approx -2h\frac{4\pi}{3}R^3 + 8\pi J R^2 \tag{106}$$

so the critical droplet and its barrier height are given by,

$$R_c = \frac{2J}{h}; \quad and \quad E_{barrier} = \frac{8\pi}{3}J(\frac{2J}{h})^2 \tag{107}$$

At small field the critical droplet size is large and the nucleation barrier is high. As the field increases, the critical droplet size is reduced until it goes to zero at the spinodal line.

Perhaps the biggest nucleation process of them all is the nucleation of the symmetry broken vacuum that occured in the early universe. The following is taken from: E. Papantonopoulos (Ed.) "The Physics of the Early Universe": The $SU(3)_C SU(2)_L U(1)_Y$ Standard Model of Strong and Electroweak interactions incorporates the concept of Spontaneous Symmetry Breaking according to which, although the Laws of Nature are symmetric under a given (local) gauge symmetry, the vacuum state is not. As a result, the vacuum expectation values of certain operators in the theory violate the symmetry. The way this is achieved in the Standard Model is through the vacuum expectation value of a scalar (Higgs) field that is an $SU(2)_L$ -doublet and carries weak hypercharge. In the broken $SU(3)_C U(1)_{em}$ vacuum three out of the four gauge bosons ($W_\pm, Z_0$) of $SU(2)_L U(1)_Y$ obtain a mass, while the fourth (photon) remains massless, corresponding to the intact electromag- netic $U(1)_{em}$ gauge interaction. In the Early Universe matter corresponds to a system in thermodynamic equilibrium with a heat bath. The thermodynamics of this system is described by the Hamiltonian of the $SU(3)_C SU(2)_L U(1)_Y$ gauge field theory. The vacuum energy of the system is determined by the minimization of the Free Energy, roughly corresponding to the so-called Effective Potential (like Landau theory), which depends on the temperature. At very high temperatures, the global vacuum state is the symmetric one, in contrast to low temperatures, where the global vacuum is the broken one. As the Universe cools down during the Radiation-Dominated epoch it makes a transition from the high temperature symmetric phase to the broken low temperature phase, or it undergoes a phase transition. This behaviour is in agreement with what happens in certain condensed matter systems. For example, a ferromagnet, when heated loses its magnetism, while at zero temperature it is characterized by a non-vanishing magnetization that breaks rotational symmetry. A more appropriate analogue is that of the phase transition from water to ice. Normally, the water-ice phase transition occurs at the freezing point of 00 C. Nevertheless, undisturbed pure water supercools to a temperature lower than the freezing point before it transforms into ice. When the transition finally occurs, after the supercooling period, the Universe is reheated due to the release of the false vacuum latent heat. Depending on the details of the theory, symmetry breaking will occur via a first order phase transition in which the field tunnels through a potential barrier, or via a second order phase transition in which the field evolves smoothly from one state to the other. A nice recent introductory review of spontaneous symmetry breaking of the vacuum by the Higgs mechanism is: “Eyes on a prize particle” Luis lvarez-Gaum and John Ellis, Nature Physics 2011. http://www.nature.com/nphys/journal/v7/n1/full/nphys1874.html

Another recent application in astrophysics is in theory for nucleation of quark matter in the core of massive neutron stars, leading to the formation of “quark stars”. Google ”Effects of quark matter nucleation on the evolution of proto-neutron stars” for the details.

A. From statistical dynamics to transport theory

Here we show that an mean field transport equation called the Boltzmann transport equation contains several of the foundations of transport theory, including the Vlasov equation for plasmas, the drift-diffusion equatons for device simulations and the equations of hydrodynamics.

Transport theory encompasses a wide range of equilibrium, close to equilibrium and far from equilibrium processes. Electronic, thermal and mass transport in plasma, gas, liquid and solid phases are included. Theories such as hydrodynamics, electronic device equations, magnetohydrodynamics form the foundations of studies of transport across all fields of science and technology. Transport may be treated at the individual particle level such as neutron transport in solids or electron transport in gas chambers, or within continuum models. A distinction is made between quantum transport processes where coherent processes dominate and classical transport where scattering processes are incoherent. There are various approximations in between for example a common treatment of electronic transport takes into account the Fermi statistics of electrons and holes but not the coherence effects, so it is not fully quantum but quantum effects are very important.

In general transport theory can be derived from the N-particle dynamics of the system, and for the systems we are treating in this course transport equations can be derived from the Liouville equation that describes the N particle phase space density $\rho(\{\vec{x}_i\}, \rho(\{\vec{}_i\}, t)$, for a system with a given interaction potential. In a similar way the quantum Lioiville equation, Heisenberg equation of motion or N-particle Schrödinger equation are the basis of dynamics in quantum systems. In general these N-particle equations are intractable so they are reduced to simpler forms by making various approximations. Here we go through a brief discussion of the Boltzmann transport equation that reduces the full N-particle phase space evolution in a classical system to an effective one particle “Boltzmann” distribution function $f(t, \vec{r}, \vec{v})$, which is the average probability that a particle is a position $\vec{r}$, with velocity $\vec{v}$ at time t. In principle, the exact Boltzmann transport equation can be found from the Liouville equation by integrating out all of the particle positions and velocities except one, however this is in generatl intractable. At equilibrium in a homogeneous system where there is no dependence of the velocities on position f reduces to the Maxwell-Boltzmann distribution of velocities.

Molecular dynamics (MD) simulations produce N-particle configurations and from these configurations, the Boltzmann probability can be calculated so we can watch the evolution of the velocity distribution as it relaxes toward equilbrium. Provided the interatomic potential is accurate, MD simulations also provide dynamical information both close to equilibrium and away from equilibrium. In contrast the Metropolis Monte Carlo (MC) method is a sampling method that is strictly valid only at equilibrium. Nevertheless dynamical rules can be added to MC to simulation equilibrium and non-equilibrium phenomena. The Boltzmann transport equation is an approximation to the phase space density and instead of working in the $3N$ dimensional phase space of the Liouville equation, it works instead in a six dimensional phase space corresponding to a one particle distribution function $f(\vec{r}, \vec{p}, t)$.

Boltzmann derived an equation for the one particle distribution function by noting that,

$$f(t+\delta t, \vec{x}+\delta\vec{x}, \vec{v}+\delta\vec{v}) - f(t, \vec{x}, \vec{v}) \rightarrow = \frac{\partial f}{\partial t}|_{collision}\delta t \tag{108}$$

The right hand side is where the interesting particle-particle interaction effects are and is called the “collision integral”.

Using a first order Taylor expansion and dividing through by δt gives,

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} f + \vec{a}\cdot\vec{\nabla}_{\vec{v}} f = \frac{\partial f}{\partial t}|_{collision} \tag{109}$$

The right hand side contains all of the information about particle-particle interactions so it is the most difficult. Below we will go through some of the properties of the collision term. However it is worth noting immediately that there are two cases where we don't treat the details of the collision term, but nevertheless the theory is very important. The first case is where the collision term is neglected completely so we have a collisionless problem. In this limit the above equation is called the Vlasov equation. This is used in studies of collisionless plasmas that occur quite frequently in astrophysics and sometimes in fusion studies. Despite the lack of particle-particle interactions many problems where the Vlasov equation is used have time varying electromagnetic fields, so the Vlasov equation is solved in conjunction with Maxwell's equations, the so-call Vlasov-Maxwell problem. A second case that is relatively simple and used frequently is to assume that the probability $f(t,\vec{r},\vec{v})$ is quite close to Maxwell-Boltzmann form, and that the relaxation toward equilbrium has has a characteristic timescale τ. In that case we have,

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}} f + \vec{a}\cdot\vec{\nabla}_{\vec{v}} f = -\frac{f-f_0}{\tau} \qquad \textit{relaxation time approximation} \tag{110}$$

A more general procedure is to consider classical scattering in the case where the interactions are the same, statistically, at any point in space. Consider two particles with momenta p_1, p_2 (where $p = mv$) scattering to so that the outgoing particles have new momenta $p_3, p4$. We introduce a two particle distribution function such that,

$$\frac{\partial f^{(2)}(p_1,p_2,t)}{\partial t} = \int d^3r_3 d^3r_4 d^3p_3 d^3p_4 W(p_1,p_2;p_3,p_4)[f^{(2)}(p_3,p_4,t) - f^{(2)}(p_1,p_2,t)] \tag{111}$$

Note that for simplicity of notation we have omitted explicit reference to the spatial dependence, and the vector nature of the momenta is also omitted for notational convenience. In the case of elastic scattering, the integral has to preserve momentum and energy conservation, while for inelastic scattering only momentum conservation is preserved. In the case of elastic scattering we have,

$$W(p_1,p_2;p_3,p_4) = w(p_1,p_2;p_3,p_4)\delta(p_1+p_2-(p_3+p_4))\delta(e_1+e_2-(e_3+e_4)) \tag{112}$$

The one particle distribution function is found by integrating the two particle function,

$$f(p_1,t) = \int f^{(2)}(p_1,p_2,t)d^3p_2 \tag{113}$$

Now we make a critical "mean field" approximation that the two particle distribution can be written as a product of one particle functions.

$$f^{(2)}(p_1,p_2,t) = f(p_1,t)f(p_2,t) \tag{114}$$

This assumes that two body collisions between uncorrelated particles are the dominant interaction mechanism, which is the "Stosszahl Ansatz" or molecular chaos assumption. Then the Boltzmann equation is then,

$$\frac{\partial f(p_1,t)}{\partial t}|_{collision} = \int dp_2^3 dp_3^3 dp_4^3 W(p_1,p_2;p_3,p_4)[f(p_3,t)f(p_4,t) - f(p_1,t)f(p_2,t)] \tag{115}$$

In quantum statistics are important the density of quantum particles also depends on the number of available states. The probability $f(p_1,t)$ is interpretted as the probability that a quantum state with quantum number p_1 is occupied at time t. The scattering probability then must take into account whether there is an available state for the scattering process, so we make the replacment,

$$\frac{\partial f(p_1,t)}{\partial t}|_{collision} \to \int dp_2^3 dp_3^3 dp_4^3 W(p_1,p_2;p_3,p_4)[f_3f_4(1-f_1)(1-f_2) - (1-f_3)(1-f_4)f_1f_2] \tag{116}$$

where the notation $f_i = f(p_i,t)$ has been introduced.

The steady state distribution for f for the classical case for elastic scattering is found by setting

$$f(p_3,t)f(p_4,t) - f(p_1,t)f(p_2,t) = 0 \tag{117}$$

with the constraints of energy and momentum conservation. Using these constraints it can be shown that this equation is satisfied by the Maxwell-Boltzmann distribution.

$$f(p,t) = Cexp[-m(\vec{v}-\vec{V})^2/2k_BT] \tag{118}$$

This shows that the Maxwell-Boltzmann distribution is a steady state solution of the dynamics and provides strong support for the equilibrium calculation given earlier. Note that this states that the average fluid velocity $\vec{V}$ is decoupled from the fluctuating component $v-\vec{V}$, so the fluctuations are Galilean invariant. Within the relaxation time approximation,

$$\frac{\partial f}{\partial t}|_{collision} \to -\frac{(f-f_0)}{\tau} \tag{119}$$

where τ is the relaxation time the function f_0 is taken to be the MB form above where a spatial dependence is introduced,

$$f_0(\vec{r},\vec{p},t) = C(\vec{r},t)exp[-m(\vec{v}-\vec{V}(\vec{r},t))^2/2k_BT(\vec{r},t)]. \tag{120}$$

As noted above, the Boltzmann equation contains many important transport equations as special cases. One example is the semiconductor device equations that describe the currents and voltages in semiconductor devices where drift and diffusion contribute to the currents. They are used to model devices such as detectors, solar cells, transistors, light-emitting diodes etc. The basic equations derived from the Boltzmann equation are the drift-diffusion equations for the electron and hole currents,

$$\vec{J}_n = en\mu_n\vec{E} + eD_n\nabla n; \quad \vec{J}_p = ep\mu_p\vec{E} - eD_p\nabla p \tag{121}$$

where e is the elementary charge, $\vec{E}$ is the electric field, n,p are the electron, hole number densities, μ_n,μ_p are the electron, hole mobilities and D_n,D_p are the electron, hole diffusion constants. These equations are solved in combination with the continuity equations for electrons and holes and the Poisson's equation for the electrostatics. The drift diffusion-equations above are derived from the steady state Boltzmann equation,

$$\vec{v}\cdot\vec{\nabla}_{\vec{r}}f - \frac{e}{m}\vec{E}\cdot\vec{\nabla}_{\vec{v}}f = -\frac{(f-f_0)}{\tau} \tag{122}$$

For example in the case of electron tranport, the current is found by integrating over the velocity distribution,

$$\vec{J}(\vec{r}) = -e\int \vec{v}f(\vec{r},\vec{v})d^3v \tag{123}$$

where the minus sign is due to the fact that the current is in the opposite direction to the velocity of the electrons. We also have,

$$n(x) = \int fd^3v; \quad \int v_x^2 fd^3v = \frac{k_BT}{m}n(x) \tag{124}$$

We can choose any axis to be the direction of the electric field and hence of the current flow, and we take this to be the x direction. Now we multiply by v_x and integrate over v_x,

$$\int v_x^2\frac{\partial f}{\partial x}dv_x - \frac{eE_x}{m}\int v_x\frac{\partial f}{\partial v_x}dv_x = \frac{1}{\tau}[\int v_xf_0dv_x - \int v_xfdv_x] = -\frac{1}{\tau}\int v_xfdv_x \tag{125}$$

where we use $\int f_0dv_x = 0$ Integrating by parts, we have,

$$\int v_x\frac{\partial f}{\partial v_x}dv_x = [v_xf]_{-\infty}^{\infty} - \int fdv_x \tag{126}$$

where the first term is zero as f goes to zero exponentially at large velocities. We then have,

$$\frac{\partial}{\partial x}[\int v_x^2\frac{\partial f}{\partial x}dv_x] + \frac{eE_x}{m}\int fdv_x = -\frac{1}{\tau}\int v_xfdv_x \tag{127}$$

Now we integrate over v_y and v_z,

$$\frac{\partial}{\partial x}[\int v_x^2 f d^3v] + \frac{eE_x}{m}\int f d^3v = -\frac{1}{\tau}\int v_x f d^3v \tag{128}$$

Now we use and use Eqs. (122) to find,

$$\frac{k_BT}{m}\frac{\partial n(x)}{\partial x} + \frac{eE_x}{m}n(x) = \frac{1}{e\tau}J_x \tag{129}$$

Using $\mu = e\tau/m$, $\mu = eD/k_BT$ leads to the first of Equations (119).

We also note that hydrodynamics can be derived from the Boltzmann equation. This starts by considering quantities that are conserved in the collisions. Then if $\chi(\vec{r},\vec{v},t)$ denotes a quanity that is conserved in the collisions we can write,

$$\int d^3v\ \chi(\vec{r},\vec{v},t)\ [\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}_{\vec{r}}f + \vec{a}\cdot\vec{\nabla}_{\vec{v}}f] = 0 \tag{130}$$

where the collision integral on the right hand side is zero because of the conservation law. From this expression, after some work and using integration by parts, we get the equations of non-viscous hydrodynamics.

$$\frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho = -\rho\nabla\cdot\vec{v}$$

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\nabla P$$

$$\frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u = -(P+u)\nabla\cdot\vec{v}$$

where P is the pressure, $u = U/V$ is the energy density and ρ is the mass density. The first equation comes from mass conservation, the second from momentum conservation and the third from energy conservation. The Navier-Stokes equations add viscosity terms to these equations.

Assigned problems and sample quiz problems

Sample Quiz Problems

Quiz Problem 1. Write down the BCS gap equation at finite temperature and compare it to the zero temperature expression. Explain the origin of the differences between the two expressions.

Quiz Problem 2. Make a plot of the BCS gap as a function of temperature. Write down the scaling behavior of the gap on approach to the critical temperature.

Quiz Problem 3. Starting from the quantum mechanical expression for the current, show that in neutral superfluids circulation is quantized. Give an expression of the quantum of circulation.

Quiz Problem 4. Starting from the quantum mechanical expressionf for the current in the presence of a vector potential, show that flux quantized. Give an expression for the flux quantum.

Quiz Problem 5. Describe the physical meaning of the healing length (ξ) in superfluids and superconductors. By considering the linearized Ginzburg-Landau equation in zero field find a solution describing the attenuation of superconducting pair density near the surface of a superconductor.

Quiz Problem 6. Using either London's original argument or starting with the Ginzburg-Landau equation, derive the London differential equation describing the penetration of parallel magnetic field into a superconducting surface.

Show that it has the solution $B(x) = B_0 e^{-x/\lambda}$, where λ is the London penetration depth.

Quiz Problem 7. What is the mixed phase of a type II superconductor? Give a physical reasoning to explain why the mixed phase of a type II superconductor can have, at sufficienty high external field, a lower free energy than the Meissner state.

Quiz Problem 8. Using London's theory, find an expression for the lower critical field of a type II superconductor.

Quiz Problem 9. Using G-L theory find an expression for the upper critical field of a type II superconductor.

Quiz Problem 10. Write down the Langevin equation for motion of a particle in the presence of random forces. Describe the physical meaning of the random forces and the meaning of Gaussian white noise.

Quiz problem 11. Write down the non-conserved order parameter relaxation equation for the Ising model within Landau theory and show that near the critical point the relaxation time diverges.

Quiz problem 12. Explain the difference between conserved dynamics and non-conserved dynamics. Give an example of the two cases.

Quiz problem 13. Sketch the phase diagrams of the van der Waals gas and the ferromagnetic Ising model, indicating the spinodal lines. How are the spinodal lines defined in the two cases.

Quiz problem 14. Explain the physical basis for the concept of a critical droplet in the homogeneous nucleation of a new stable phase from a metastable phase. Find an expression for the critical droplet size for an Ising model in a magnetic field at low temperature.

Quiz problem 15. Derive the Vlasov equation for the time evolution of the single particle distribution function $f(t, \vec{r}, \vec{v})$ for a collisionless system.

Assigned problems

Assigned Problem 1. By doing a variation with respect to ψ^* of the Ginzburg-Landau free energy (Eq. (64)) of the notes, derive the Ginzburg-Landau equation (65).

Assigned Problem 2. By doing a variation with respect to the vector potential A of the Ginzburg-Landau free energy (Eq. (64)) of the notes, derive the expression for the current (66).

Assigned Problem 3. Show that minimizing the Helmholtz free energy within London theory,

$$f_1 = \frac{1}{2\mu_0} \int_0^\infty (B^2 + \lambda^2 \mu_0^2 J^2) 2\pi r dr \tag{131}$$

gives the London equation.

Assigned Problem 4. Using London theory, find the Helmholtz free energy per unit length of a vortex in a superconductor in an external field H.

Assigned Problem 5. Using the Gibb's free energy within London theory, demonstrate that the triangular lattice vortex array has lower energy than the square lattice array, for a fixed applied field, $H > H_{c1}$, which is close to H_{c1}.

Assigned Problem 6. Go through the calculations to show that the kinetic energy and angular momentum per unit length of a quantum of circulation in a neutral superfluid are given by equations (40) and (41) of the notes.

Assigned Problem 7. Considering only the kinetic energy of the supercurrent and using arguments like those of Problem 6, show that the supercurrents circulating around a flux quantum in a superconductor have energy approximately given by Eq. (55) of the notes.

Assigned Problem 8. Derive the drift diffusion equation

$$\vec{J}_n = en\mu_n\vec{E} + eD_n\frac{\partial n}{\partial x}; \quad \vec{J}_p = ep\mu_p\vec{E} - eD_p\frac{\partial p}{\partial x} \tag{132}$$

from the Boltzmann equation using the relaxation time approximation.

Assigned Problem 9. The collision integral (Eq. (113) of the notes) is zero when,

$$f(\vec{p}_1)f(\vec{p}_2) = f(\vec{p}_3)f(\vec{p}_4). \tag{133}$$

Show that the Maxwell Boltzmann distribution

$$f(t, \vec{r}, \vec{v}) = Ce^{-\alpha(\vec{v}-\vec{V})^2} \tag{134}$$

satisfies this equation, for the case of elastic scattering.