PHY831 - Midterm 2, Monday October 22nd 2012
Answer all questions. Time for midterm - 50 minutes

Name:

Formulae that might be helpful

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = (\frac{\pi}{a})^{1/2}e^{\frac{b^2}{4a}} \tag{1}$$

Relations between partial derivatives,

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z ; \quad \text{Inversion} \tag{2}$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z ; \quad \text{Addition of a variable} \tag{3}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 ; \quad \text{Triple Product} \tag{4}$$

If $A(x,y)$, then,

$$\left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_y + \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z ; \quad non-natural\ derivative \tag{5}$$

Response functions are defined as follows;

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = \left(\frac{\partial U}{\partial S}\right)_{V,N} \left(\frac{\partial S}{\partial T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N}, \tag{6}$$

$$C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial S}\right)_{P,N} \left(\frac{\partial S}{\partial T}\right)_{P,N} = T\left(\frac{\partial S}{\partial T}\right)_{P,N}, \tag{7}$$

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N} = -\left(\frac{\partial lnV}{\partial P}\right)_{S,N}, \tag{8}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = -\left(\frac{\partial lnV}{\partial P}\right)_{T,N}, \tag{9}$$

$$\alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} = \left(\frac{\partial lnV}{\partial T}\right)_{P,N}, \tag{10}$$

For any s we have,

$$\int_0^{\infty} \frac{x^{s-1}dx}{e^x - 1} = \Gamma(s)\zeta(s), \tag{11}$$

where $\Gamma(s) = (s-1)!$ for s a positive integer, and $\zeta(2) = \pi^2/6$, $\zeta(3) = 1.202...$, $\zeta(4) = \pi^4/90$.
For n even we have,

$$\int_{-\infty}^{\infty} dt \frac{t^n e^t}{(e^t+1)^2} = 2n(1-2^{1-n})(n-1)!\zeta(n) \tag{12}$$

while for odd n this integral is zero.

$$\hbar = 1.055 \times 10^{-34} Js;\ k_B T_{room} = .025eV;\ e = 1.602 \times 10^{-19}C;\ k_B = 1.38 \times 10^{-23} J/K;\ m_e = 9.11 \times 10^{-31} kg \tag{13}$$

1. (15 points)

N is the total number of Krypton atoms in a chamber where the surfaces of the chamber are lined with graphite. Some of the Kr atoms are in the gas phase and others are bound to the graphite surface. If there are a total of N_B sites on the graphite surfaces of the chamber where Kr atoms can bind, with binding energy ϵ (i.e. the bound state is lower in energy), find an expression for the average number of Kr atoms that are bound to the surfaces of the chamber as a function of temperature. Take the volume of the chamber to be V, ignore the interactions between Krypton atoms and consider the case where $N \leq N_B$. Find an explicit solution in the limit where the number of Kr atoms bound to the surface is small. Under what conditions does this limit hold?

Solution

We assume that the particles can be treated as classical particles. If there are N_s particles bound to the graphite surface, their Helmholtz free energy is given by,

$$F = U - TS = -N_s\epsilon - k_BTln(\frac{N_B!}{N_s!(N_B - N_S)!}) = -N_s\epsilon + k_BTN_B[xln(x) + (1-x)ln(1-x)] \tag{14}$$

where $x = N_s/N_B$. The chemical potential of the particles at the surface is given by,

$$\mu_s = \frac{\partial F}{\partial N_s} = -\epsilon + k_BTln(x/(1-x)). \tag{15}$$

Particles in the gas phase have chemical potential,

$$\mu_g = k_BTln(\frac{N_g\lambda^3}{V}). \tag{16}$$

At equilibrium the two chemical potentials must be the same $\mu_s = \mu_g$, and using $N_g = N - N_s$ we find,

$$k_BT[ln(N - N_s) + ln(\frac{\lambda^3}{V})] = -\epsilon + k_BTln(x/(1-x) \tag{17}$$

Consider the limit where there are not many bound atoms so that $N_s/N \to 0$ and $N_s/N_B \to 0$ so that,

$$\frac{N_s}{N} \approx \frac{N_B\lambda^3}{V}e^{\beta\epsilon} \tag{18}$$

Even at high temperatures atoms are bound to the surface - and the number that are bound depends on the relative entropy of the surface atoms and gas atoms, as well as the binding energy.

An alternative approach that gives the same final result is to write down the partition function of the whole system,

$$Z = \sum_{N_s=0}^{N} Z_{surface}(N_s)Z_{gas}(N - N_s) \tag{19}$$

and hence $F = -k_BTln(Z)$. Then we can find the optimal value of N_s, by solving $\partial F/\partial N_s = 0$. This leads to the same equation as $\mu_s = \mu_g$ above.

2. (15 points)

Recent observations indicate that the universe is expanding at an increasing rate so there is an intrinsic acceleration of the size of the universe and the Hubble constant is actually getting larger. This observation led to the 2011 Nobel prize in physics (Perlmutter, Schmidt, Reiss). Consider the universe to be a container that has size L and volume L^3.

(i) Considering the cosmic microwave background (CMB) radiation to be a ideal Bose gas of photons, find an expression for the pressure that this gas exerts on the universe.

(ii) Now consider a model for pressure due to dark matter. Treat dark matter as being due to a single type of non-relativistic spinless fermion of mass m in its ground state. Take the total mass of the universe to be M. Astronomical measurements suggest that 84% of this mass is dark matter. Find an expression for the pressure due to this fermionic dark matter.

(iii) Make order of magnitude estimates of the CMB and dark matter pressures, taking the mass of the dark matter fermions to be that of the electron. Which pressure is likely to be larger? Take the mass of the universe to be $10^{55}kg$, and the size of the universe to be $10^{26}m$.

Solution

(i) The internal energy of the photon gas is given by,

$$U = 2(\frac{L}{2\pi})^3 \int_0^\infty 4\pi k^2 dk(\hbar kc)\frac{e^{-\beta\hbar kc}}{1-e^{-\beta\hbar kc}} \tag{20}$$

using $x = \beta\hbar kc$ this reduces to,

$$\frac{U}{V} = \frac{1}{\beta\pi^2}\frac{1}{(\beta\hbar c)^3}\int_0^\infty \frac{x^3 dx}{e^x - 1} \tag{21}$$

The pressure is given by,

$$P = \frac{1}{3}\frac{U}{V} = \frac{1}{\pi^2\hbar^3 c^3}\frac{\pi^4}{45}(k_B T)^4 \tag{22}$$

which is the pressure exerted by the photon gas on the surface of the universe. The pressure depends only on the temperature and fundamental constants.

(ii) The internal energy of the degenerate fermi gas is given by,

$$U = (\frac{L}{2\pi})^3 \int_0^{k_F} 4\pi k^2 \frac{\hbar^2 k^2}{2m}; \tag{23}$$

we also define,

$$N = (\frac{L}{2\pi})^3 \frac{4\pi}{3}k_f^3; \quad \epsilon_F = \frac{\hbar^2 k_F^2}{2m} \tag{24}$$

so that,

$$U = \frac{3}{5}N\epsilon_F \tag{25}$$

and

$$P = \frac{2}{3}N\epsilon_F = \frac{2}{5}\frac{\hbar^2}{2m}(6\pi^2)^{2/3}\rho^{5/3} \tag{26}$$

where $\rho = N/V$.

(iii) For the CMB at $T = 2.713K$, we have,

$$P = \frac{1}{\pi^2\hbar^3 c^3}\frac{\pi^4}{45}(k_B T)^4 \approx \frac{1}{10}10^{34*3}\frac{1}{27}10^{-24}\frac{81}{45}(1.38\times 2.713)^4(10^{-23*4}) \approx 2\times 10^{-14}N/m^2 \tag{27}$$

For dark matter fermions of mass m we have,

$$P = \frac{2}{5}\frac{\hbar^2}{2m}(6\pi^2)^{2/3}\rho^{5/3}. \tag{28}$$

We keep m as a free parameter so that $N = M_{dark}/m = 0.84\times 10^{55}/m$ and $\rho \approx N/V = 10^{55}/m \times 10^{-26\times 3}$. we then find,

$$P = \frac{1}{5}(10^{-68})(16)(10^{-40})\frac{1}{m^{8/3}} = 3\times 10^{-108}\frac{1}{m^{8/3}} \tag{29}$$

where the mass m is in kg. If the dark matter fermions have a mass of order that of the electron, $m = 9.11\times 10^{-31}$, then the dark matter pressure is roughly $10^{-28}N/m^2$. However since the formula has a factor $1/m^{8/3}$, if the mass of the dark matter fermions is lower, then it is possible for them to generate a pressure which is large. For example the current limit on the mass of the electron neutrino is not precise but is around $m < 0.5eV/c^2$ as compared to the electron mass which is $0.511MeV/c^2$. If we use the mass of the electron neutrino in our calculation (Eq. (29)), the pressure is 10^{16} times larger, and may be larger than that of the CMB.