PHY831 - Midterm I, Monday September 26th 2011

Answer all questions. Time for midterm - 50 minutes

Name:

Formulae that might be helpful

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2+bx} = (\frac{\pi}{a})^{1/2} e^{\frac{b^2}{4a}}; \qquad \int_0^{\infty} dx\ x^n e^{-ax^2} = \frac{1}{2a^{(n+1)/2}} \Gamma(\frac{n+1}{2}) \tag{1}$$

$$\int_0^{\infty} \frac{x^{s-1} dx}{e^x - 1} = \Gamma(s)\zeta(s), \tag{2}$$

where $\Gamma(s) = (s-1)!$ for s a positive integer, and $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202...$, $\zeta(4) = \pi^4/90$.

Response functions are defined as follows;

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = \left(\frac{\partial U}{\partial S}\right)_{V,N} \left(\frac{\partial S}{\partial T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N}, \tag{3}$$

$$C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial S}\right)_{P,N} \left(\frac{\partial S}{\partial T}\right)_{P,N} = T\left(\frac{\partial S}{\partial T}\right)_{P,N}, \tag{4}$$

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N} = -\left(\frac{\partial lnV}{\partial P}\right)_{S,N}, \tag{5}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = -\left(\frac{\partial lnV}{\partial P}\right)_{T,N}, \tag{6}$$

$$\alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} = \left(\frac{\partial lnV}{\partial T}\right)_{P,N}, \tag{7}$$

1. (10 points)
White general expressions $ln(\Xi)$, PV, N/V and U for the Fermi gas are valid in any dimension for the dispersion relation $\epsilon_p = p^2/2m$. For the two dimensional case, find an expression for the pressure in the high temperature limit and in the ground state.

Solution. Either the integrals,

$$\frac{P}{k_B T} = \frac{c_d}{h^d}\int_0^\infty dp\ p^{d-1} ln\left(1 + ze^{-\beta p^2/2m}\right) \tag{8}$$

$$\frac{N}{L^d} = \frac{c_d}{h^d}\int_0^\infty dp\ p^{d-1}\frac{ze^{-\beta p^2/2m}}{1 + ze^{-\beta p^2/2m}} \tag{9}$$

$$\frac{U}{L^d} = \frac{c_d}{h^d}\int_0^\infty dp\ p^{d-1}\frac{p^2}{2m}\frac{ze^{-\beta p^2/2m}}{1 + ze^{-\beta p^2/2m}} \tag{10}$$

and/or

$$\frac{P}{k_B T} = \frac{1}{\lambda^d} f_{d/2+1}(z); \qquad \frac{N}{L^d} = \frac{1}{\lambda^d} f_{d/2}(z); \qquad \frac{U}{L^d} = \frac{d}{2}\frac{k_B T}{\lambda^d} f_{d/2+1}(z) \tag{11}$$

At high temperatures the behavior is like that of a classical gas so in two dimensions, $PA = Nk_B T$, where A is the area. Note that pressure in two dimensions is force per unit length. In the ground state we have a degenerate Fermi system, with Fermi wavevector,

$$N = (\frac{L}{2\pi})^2 \pi k_F^2 \quad \text{ so that } \quad k_F = (\frac{4\pi N}{L^2})^{1/2} \tag{12}$$

The energy, U and Fermi energy E_F are given by,

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2 4\pi N}{2mL^2};$$

and

$$U = (\frac{L}{2\pi})^2 \int_0^{k_F} 2\pi k dk \frac{\hbar^2 k^2}{2m} = (\frac{L}{2\pi})^2 2\pi \frac{\hbar^2}{2m}\frac{k_F^4}{4} = \frac{1}{4}(\frac{L}{2\pi})^2 2\pi E_F(\frac{4\pi N}{L^2}) = \frac{1}{2}NE_F \tag{15}$$

The pressure is found from the relation,

$$PV = \frac{s}{d}U; \quad \text{ so that } \quad P = \frac{1}{2}\frac{N}{L^2}E_F \tag{16}$$

2. (10 points)
White general expressions $ln(\Xi)$, PV, N/V and U for the Bose gas are valid in any dimension for the dispersion relation $\epsilon_p = p^2/2m$. For the three dimensional case, find an expression for the pressure in the high temperature limit and in Bose condensed state.

Solution. We have,

$$\frac{P}{k_B T} = -\frac{4\pi}{h^3}\int_0^\infty dp\ p^2 ln\left(1 - ze^{-\beta p^2/2m}\right) - \frac{1}{V}ln(1-z); \quad \text{Bose gas} \tag{17}$$

$$\frac{N}{V} = \frac{4\pi}{h^3}\int_0^\infty dp\ p^2 \frac{ze^{-\beta p^2/2m}}{1 - ze^{-\beta p^2/2m}} + \frac{1}{V}\frac{z}{1-z}; \quad \text{Bose gas} \tag{18}$$

and

$$\frac{U}{V} = \frac{4\pi}{h^3}\int_0^\infty dp\ p^2 \frac{p^2}{2m}\frac{ze^{-\beta p^2/2m}}{1 - ze^{-\beta p^2/2m}}; \quad \text{Bose gas} \tag{19}$$

and

$$P = \frac{k_B T}{\lambda^3} g_{5/2}(z) - \frac{k_B T}{V} ln(1-z); \qquad \frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z) + \frac{1}{V}\frac{z}{1-z}; \qquad \frac{U}{V} = \frac{3}{2}\frac{k_B T}{\lambda^3} g_{5/2}(z) \tag{20}$$

In the high temperature limit $PV = Nk_B T$. In the low temperature limit,

$$P = \frac{k_B T}{\lambda^3} g_{5/2}(1); \quad where g_{5/2}(1) = \zeta(5/2) = \sum_{l=1}^{\infty} \frac{1}{l^{5/2}} \tag{21}$$

3. (10 points)
(a) Derive the Planck blackbody spectral density $u(\omega)$ and find the behavior of its maximum as a function of temperature.
(b) Integrate this function to find the energy density in the blackbody spectrum.
(c) What is the relation between the energy density and the Stefan-Boltzmann equation? Explain this relation.

Solution. The blackbody energy spectrum is modeled as the internal energy of a photon gas at temperature T, given by,

$$U = 2\left(\frac{L}{2\pi\hbar}\right)^3 \int_0^{\infty} 4\pi p^2 dp \; (pc) \frac{e^{-\beta pc}}{1-e^{-\beta pc}} = V\int d\omega \; u(\omega) \tag{22}$$

where $n(\omega)$ is the number density of photons and $u(\omega)$ is the energy density at angular frequency ω. The additional factor of two in front of this equation is due to the two polarizations that are possible for the photons. a) Using $p = \hbar\omega/c$, we find the blackbody spectral energy density

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}. \tag{23}$$

b) Integration using the formula on the cover sheet yields

$$\frac{U}{V} = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} T^4; \quad PV = \frac{1}{3}U; \quad N = V\frac{2\zeta(3)(k_B T)^3}{\pi^2 h^3 c^3} \tag{24}$$

c) The intensity from a blackbody is given by the Stefan-Boltzmann law,

$$I = \sigma T^4 = \frac{c}{4}\frac{U}{V} \tag{25}$$

The factor c comes from the relationship between the energy of a travelling wave and its intensity, and the second is a geometric factor due to an assumption of isotropic emission from a small surface element on the surface of the emitter. To understand the first factor, consider a classical EM wave in free space with energy density $u = \epsilon_0 E_0^2/2 +_0^2/(2\mu_0)$. In the direction of propagation of the wave, the energy crossing a surface of area A per unit time is,

$$Energy\ per\ unit\ time = Power = u * A * c \quad \text{so that} \quad I_w = Power/Area = uc \tag{26}$$

where I_w is the intensity of the wave. This applies to both the peak and rms intensity of the wave, provided the energy density is the peak or rms value respectively. The geometric factor comes from considering a small flat surface element that emits radiation in all directions. In the case of blackbody radiation, this element is considered to be at the surface, so it emits half of its radiation back into the black body and half out of the black body. In addition, the radiation in the direction normal to the surface is reduced from the total radiation emitted from the surface element due to the assumption of isotropic emission. The component normal to the surface is found by finding the component of the electric field in the formal direction, $E_0 cos(\theta)$, then squaring this to get the correct projection of the intensity, and then averaging over angles θ in a hemisphere. The result is that we need to average $cos^2(\theta)$ over a half period. This leads to a geometric factor of 1/2. Multiplying these two factors of 1/2 gives the total geometric factor of 1/4. In most applications, the Stefan-Boltzmann law needs to be modified to account for the emissivity of the material (e) and the geometry of the surface and the location of the observer with respect to the surface, if the surface is not spherical.