PHY831 - Midterm IV, Wednesday December 7th 2011

Answer all questions. Time for midterm - 30 minutes

Name:

$$\int_{-\infty}^{\infty} dx\ e^{-ax^2+bx} = (\frac{\pi}{a})^{1/2} e^{\frac{b^2}{4a}}; \qquad \int_0^{\infty} dx\ x^n e^{-ax^2} = \frac{1}{2a^{(n+1)/2}}\Gamma(\frac{n+1}{2}) \tag{1}$$

$$\int \frac{dx}{(1+x^2)^{1/2}} = Sinh^{-1}(x); \qquad \int_0^{\infty} \frac{x^{s-1}dx}{e^x-1} = \Gamma(s)\zeta(s), \tag{2}$$

where $\Gamma(s) = (s-1)!$ for s a positive integer, and $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.202...$, $\zeta(4) = \pi^4/90$.

$$\left(\frac{\partial x}{\partial y}\right)_z = 1/\left(\frac{\partial y}{\partial x}\right)_z; \qquad \left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial w}\right)_z \left(\frac{\partial w}{\partial y}\right)_z; \tag{3}$$

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1\ ; \qquad \left(\frac{\partial A}{\partial x}\right)_z = \left(\frac{\partial A}{\partial x}\right)_y + \left(\frac{\partial A}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_z; \tag{4}$$

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_{V,N} = T\left(\frac{\partial S}{\partial T}\right)_{V,N}; C_P = \left(\frac{\partial H}{\partial T}\right)_{P,N} = T\left(\frac{\partial S}{\partial T}\right)_{P,N}, \tag{5}$$

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}; \quad \kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}; \quad \alpha_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} \tag{6}$$

$$\frac{U}{L^d} = \frac{c_d}{h^d}\int_0^{\infty} dp\ p^{d-1}(cp^s)\frac{ze^{-\beta cp^s}}{1-ze^{-\beta cp^s}} = \frac{d}{s}\frac{k_BT}{\lambda^d}g_{1+d/s}(z); \quad P = \frac{s}{d}\frac{U}{L^d} \tag{7}$$

$$I(r,\nu\to\infty) = \int_0^{\infty} dx\ x^{r-1}\frac{ze^{-x}}{1+ze^{-x}} = \frac{1}{r}\int_0^{\infty} dx\ x^r \frac{e^{x-\nu}}{(e^{x-\nu}+1)^2} = \frac{1}{r}[\nu^r + \frac{\pi^2}{12}r(r-1)\nu^{r-2} + ...] \tag{8}$$

$$H = -\frac{1}{2}\sum_{ij} J_{ij}S_iS_j - \sum_i h_iS_i \quad \to \quad (for\ spin \pm 1)\ \ Z_{MF} = 2^N e^{-\frac{1}{2}\beta\sum_{ij}J_{ij}m_im_j}\prod_{i=1}^{N}(Cosh(\beta\sum_j J_{ij}m_j + \beta h_i)). \tag{9}$$

$$Z_{vdw} = \frac{q^N}{N!\lambda^N}; \quad \text{where} \quad q = (V-Nb)e^{aN/(Vk_BT)}; \quad \frac{Pv}{k_BT} = \sum_{l=1}^{\infty} a_l(T)(\frac{\lambda^3}{v})^{l-1} \quad (virial\ expansion) \tag{10}$$

$$g_{GL} = \int dV\left[\frac{1}{2m}|(-i\hbar\nabla - qA)\psi(\vec r))|^2 + a(T)|\psi(\vec r)|^2 + \frac{b(T)}{2}|\psi(\vec r)|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B\cdot H\right]. \tag{11}$$

$$\frac{\delta g}{\delta\psi^*} = a(T)\psi + b(T)|\psi|^2\psi + \frac{1}{2m}(-i\hbar\nabla - qA)^2\psi = 0; \tag{12}$$

$$\frac{\delta g}{\delta A} = 0 \quad \to \quad j_s = \frac{-iq\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m}A|\psi|^2 = \frac{q}{m}|\psi|^2(\hbar\nabla S - qA) = q|\psi|^2 v_s \tag{13}$$

$$H_{MF} - \mu N = \sum_{\vec k\sigma}(\epsilon_{\vec k\sigma} - \mu)\ a^\dagger_{\vec k\sigma}a_{\vec k\sigma} - \sum_{\vec k}(\Delta_{\vec k}a^\dagger_{\vec k\uparrow}a^\dagger_{-\vec k,\downarrow} + \Delta^*_{\vec k}a_{-\vec k\downarrow}a_{\vec k\uparrow} - b^*_{\vec k}\Delta_{\vec k}); \tag{14}$$

$$H_{MF} - \mu N = \sum_{\vec k}(\epsilon_{\vec k} - \mu - E_{\vec k} + \Delta_{\vec k}b^*_{\vec k}) + \sum_{\vec k} E_{\vec k}(\gamma^\dagger_{\vec k\uparrow}\gamma_{\vec k\uparrow} + \gamma^\dagger_{\vec k\downarrow}\gamma_{\vec k\downarrow}), \tag{15}$$

$$\Delta_{\vec k} = -\sum_{\vec l} V_{\vec k\vec l}\, b_{\vec l}; \quad b_{\vec k} = < a_{-\vec k\downarrow}a_{\vec k\uparrow} > = \frac{\Delta_{\vec k}}{2E_{\vec k}}(1-2f(E_{\vec k})); \quad E_{\vec k} = ((\epsilon_{\vec k} - \mu)^2 + |\Delta_{\vec k}|^2)^{1/2} \tag{16}$$

1. (10 points)

(i) Draw the phase diagram of a type II superconductor in the H-T plane. Indicate the Meissner, mixed and normal phases. Describe the physical meanings of the coherence length (ξ) and penetration depth (λ) in superconductors.

(ii) Sketch the magnetic field profile and density of superconducting electrons as a function of distance from the center of a vortex. Indicate the relevance of the lengths ξ and λ. Also sketch a top view of a vortex in a superconductor indicating the direction of the magnetic field and the direction of the screening currents.

(iii) Prove that the smallest quantum of flux that can exist in a superconductor is $\phi_0 = h/(2e)$.

Solution

(i) The coherence length, ξ, describes the length scale over which the density of superconducting electrons varies, either through random fluctuations, at a surface, or in a vortex core. The penetration depth λ describes the length scale over which magnetic fields penetrate into a superconducting domain.

(ii) The currents circulating around a vortex core act to screen the magnetic field from the interior of the superconductor.

(iii) The equation for the current in a superconductor is given by,

$$j_s = \frac{q}{m}|\psi|^2(\hbar\nabla S - qA) \tag{17}$$

If we make a loop around a vortex along a path where the current is zero, we have,

$$\int(\hbar\nabla S - qA)\cdot d\vec{l} = \hbar\Delta S - q\int \vec{A}\cdot d\vec{l} = \hbar 2\pi n - q\phi \tag{18}$$

where we use the fact that $\int A\cdot dl = \int B\cdot da = \phi$, and the fact that the phase change around the loop ΔS has to be an integer multiple of 2π to ensure that the wavefunction is single valued. The smallest quantum of flux is then,

$$\phi_0 = h/q = h/2e \tag{19}$$

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2. (10 points)

(i) Write down the scaling assumption for the magnetization and show how it leads to the exponent relations, $\beta = \Delta - \gamma$ and $\Delta = \beta\delta$.

(ii) The Lifshitz condition is based on the ratio $\xi^{-(d-2+\eta)}/o^2$, where o is the order parameter. Show that it leads to an upper critical dimension of four for the Ising model and an upper critical dimension of six for the percolation model.

Solution

(i) The scaling assumption is,

$$m \approx t^\beta m_s(h/t^\Delta) \tag{20}$$

To ensure that the behavior $m \sim h^{1/\delta}$ at $t = 0$ is recovered, we require, $m_s(y \to \infty) \sim y^{1/\delta}$, so that,

$$m \approx t^\beta (\frac{h}{t^\Delta})^{1/\delta} \sim h^{1/\delta}; \qquad \text{provided} \qquad \beta - \frac{\Delta}{\delta} = 0 \tag{21}$$

which proves that $\Delta = \beta\delta$. To find the first relation, we consider the limit of small h/t^δ, where

$$m \approx t^\beta \approx \int \chi dh \approx \int_0^{t^\Delta} t^{-\gamma} dh = t^{-\gamma+\Delta} \tag{22}$$

so that $\beta = \Delta - \gamma$.

(ii) The order parameter behaves as t^β, while the correlation length behaves as $t^{-\nu}$, which leads to

$$\frac{\xi^{-(d-2+\eta)}}{o^2} \approx t^{(d-2+\eta)\nu - 2\beta} \tag{23}$$

the Lishitz conditions takes the mean field exponents to find the critical dimension below which the ratio above diverges. It is evident that as the dimensions reduces, the exponent decreases and becomes negative below a critical dimension given by,

$$d_{uc} = 2 + 2\beta/\nu - \eta \tag{24}$$

The mean field values for the exponents $\eta = 0, \nu = 1/2$ apply to both the Ising and percolation models. The order parameter exponents for these cases is $\beta_{Ising} = 1/2$, $\beta_{perc} = 1$, so that the upper critical dimensions for the two cases are,

$$d_{uc} = 4 \quad (Ising); \qquad d_{uc} = 6 \quad (percolation) \tag{25}$$

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3. (10 points)

(i) Explain the concept of the critical droplet or cluster size in the dynamics of first order phase transitions. Illustrate the discussion by finding the critical droplet size for the ferromagnetic Ising model in an applied field, for temperatures $T < T_c$.

(ii) Qualitatively discuss the meaning of a spinodal line within mean field theory and also within the droplet picture. Sketch the location of the mean field spinodal line for the ferromagnetic Ising model on a $h - T$ phase diagram, and for the liquid-gas transition on a $P - v$ phase diagram. Indicate the regions that are metastable and unstable.

Solution

(i) The critical droplet is a first approximation to understanding the process of nucleation of a stable phase from a deeply metastable initial state. It can be applied to any system provided the correct bulk and surface energies are used. For the case of an Ising ferromagnet, consider a deeply metastable initial phase consisting of a up spin configuration at temperature well below the critical temperature. The field is assumed to be small and to favor the down state. A spherical droplet excitation from the up phase has energy,

$$E(R) = -2hc_d R^d + 2\sigma k_d R^{d-1} \tag{26}$$

where σ is the surface energy between the up and down phases. At small droplet sizes the surface energy dominates, while at large enough droplet sizes, the field energy dominates. The maximum in the droplet energy defines the exitation barrier to the stable phase. To find this energy barrier, we find the maximum of the droplet energy using,

$$\frac{\partial E}{\partial R} = 0; \qquad \text{so that} \qquad R_c = \frac{\sigma(d-1)k_d}{hdc_d} \tag{27}$$

The energy barrier is then $E(R_c)$.

(ii) Within mean field theory, the spinodal line corresponds to the point where the metastable minimum in the energy ceases to exist, so that the new phase grows spontaneously. In an Ising magnet the applied field favoring the down state can be increased enough to remove the metastablility, so that the critical droplet size goes to zero. In a van der Waals gas the condition for the spinodal lines is $\partial P/\partial v = 0$. The metastable regimes are closest to the coexistence boundary in the liquid-gas problem and closest to the zero field line in the Ising problem. If the metastable minimum ceases to exist or the droplet energy barrier goes to zero, we are in the unstable or spinodal regime.