Solutions to problems for Part 4

Sample Quiz Problems

Quiz Problem 1. Describe the physical meaning of the coherence length (ξ) in superconductors. By considering the linearized Ginzburg-Landau equation in zero field find a solution describing the attenuation of superconducting pair density near the surface of a superconductor.

Solution

The coherence length or healing length describes the length scale over which the magnitude of the superconducting order parameter, or density of superconducting pairs, is reduced near a normal surface. See Eq. (7) of the notes for the calculation.

Quiz Problem 2. Using either London's original argument or starting with the Ginzburg-Landau equation, derive the London differential equation describing the penetration of parallel magnetic field into a superconducting surface. Show that it has the solution $B(x) = B_0 e^{-x/\lambda}$, where λ is the London penetration depth.

Solution

See Eqs. (12) - (23) of the notes.

Quiz Problem 3. What is the mixed phase of a type II superconductor? Give a physical reasoning to explain why the mixed phase of a type II superconductor can have, at sufficienty high external field, a lower free energy than the Meissner state.

Solution

The mixed phase of a type II superconductor consists of flux quanta that form a triangular array, where the flux quanta consist of a normal vortex core with supercurrents circulating around the normal core. The formation of the normal core costs condensation energy $E_c \pi \xi^2$, where the condensation energy is the energy that stabilizes the superconducting state over the normal state. At zero temperature the BCS value for E_c is $N(\epsilon_F)\Delta(0)^2/2$, while within Landau theory it is $-a^2/2b$. The circulating supercurrents also cost energy, which within London theory is approximated by the kinetic energy $n_c m v_s^2/2$. This leads to the energy cost $\epsilon_1 = [\phi_0^2/(4\pi\mu_0\lambda^2)] ln(\lambda/\xi)$, which is correct in the limit $\kappa = \lambda/\xi >> 1$. The core energy and kinetic energy costs are offset by the energy gain due to the work done by the external reservoir, $\phi_0 H$ per vortex. From this calculation we find that it is energetically favorable for vortices to enter a strongly type II superconductor for fields $H > H_{c1} \approx \epsilon_1/\phi_0$.

Quiz Problem 4. Write down the scaling assumption for the magnetization and show how it leads to the exponent relations, $\beta = \Delta - \gamma$ and $\Delta = \beta\delta$.

Solution

The scaling assumption for the magnetization is,

$$m = t^{\beta} m_s(h/t^{\Delta}) \tag{1}$$

To ensure that the scaling behavior $m(0) \sim h^{1/\delta}$ is recovered, we require,

$$m_s(y) \sim y^{1/\delta}; \qquad \text{and} \qquad \beta - \frac{\Delta}{\delta} = 0 \tag{2}$$

To find the second relation, we note that $m \sim \int \chi dh$, so that,

$$m \sim t^{\beta} \approx \int_0^{t^{\Delta}} t^{-\gamma} dh \approx t^{\Delta-\gamma} \tag{3}$$

which proves the first relation.

Quiz Problem 5. Write down and explain the scaling assumption used in RG analysis of the ferromagnetic Ising phase transition. Show that if the length rescaling b is taken to be $b = \xi$, then the expected scaling behaviors are recovered.

Solution The scaling relation is,

$$m = b^{-\beta/\nu} m_s(b^{D_t}t, b^{D_h}h \tag{4}$$

If $b = \xi = t^{-\nu}$, then we have,

$$m = t^{\beta} m_s(t^{-\nu D_t}t, t^{-\nu D_h}h) \;=\; t^{\beta} m_s(1, h/t^{\Delta}) \tag{5}$$

where we used $D_t = 1/\nu, D_h = \Delta/\nu$. The scaling relation is consistent with the standard relation in Eq. (54) of the notes.

Quiz Problem 6. Explain the concept of the critical droplet or cluster size in the dynamics of first order phase transitions. Illustrate the discussion by finding the critical droplet size for the ferromagnetic Ising model in an applied field, for temperatures $T < T_c$.

Solution

Consider an Ising ferromagnetic that is prepared in an equilibrium up spin state for $T < T_c$, with $h > 0$. Now the magnetic field orientation is changed to $h < 0$, so the down spin state becomes the equilibrium orientation. In order for the up spin state to convert to the down spin state, we must nucleate the down spin phase within the up spin phase. The energy cost of forming a spherical down spin domain in an up spin sea is,

$$E(R) = -2hc_d R^d + 2\sigma k_d R^{d-1} \tag{6}$$

This "droplet" energy has a maximum at a critical droplet size found by minimizing with respect to R. For small fields the critical droplet size is large, while for large fields the critical droplet size goes to zero. The energy evaluated at the critical droplet size is the energy barrier. The point at which the energy barrier is approximately $k_B T$ is the droplet theory prediction of the spinodal point.

Quiz Problem 7. Discuss the difference between conserved order parameter (COP) and non-conserved order parameter (NCOP) dynamics. For a G-L free energy $F(m)$ for an Ising model, where m is the order parameter, write down the expressions for the COP and NCOP relaxation dynamics of the system. Which relaxation dynamics is slower? Why?

Solution

In a non-conserved order parameter dynamics, fluctuations of the system are not constrained by conservation laws, so free exchange with a reservoir is possible. An example of this type of dynamics is the dynamics of an Ising model in contact with a thermal reservoir. In contrast if there is a conservation law on the mass or other properties of the system, the dynamics must preserve the conservation laws and for this reason the dynamics is typically slower. An example of a system with a conserved dynamics is the dynamics of a liquid within the micro-canonical ensemble. If however we consider the dynamics of the liquid in the grand-cannonical ensemble were particle exchange is allowed, then the dynamics may return to the non-conserved case. The G-L equations for the two cases are given by Eq. (114) of the notes. These equations also include a noise term which enables analysis of the effect of fluctuations on the dynamics.

Quiz Problem 8. Write down the Langevin equation for the random motion of a particle. Explain the physical reasoning behind the Langevin approach to dynamics, in particular the noise term. Explain what is mean by "Gaussian white noise". Write down the expressions for the average value of the noise and the correlation function of the noise. Show that the Langevin equation leads to exponential relaxation of the velocity, provided there is no external force applied to the particle.

Solution
The Langevin equation for the motion of a particle in the presence of random forces is given by,

$$m\frac{d\vec{v}}{dt} = -\frac{\vec{v}}{B} + \eta'(t); \quad \text{where} \quad < \eta(t) >= 0; \quad < \eta(t)\eta(t') >= c\delta(t,t') \tag{7}$$

where B is the damping coefficient and $\eta'(t)$ is the random force. The random force is uncorrelated, so it's correlations are given by a delta function. If we take a time average of the Langevin equation, we find the simple result,

$$< \vec{v}(t) >= \vec{v}(0)e^{-t/\tau} \quad \text{where} \quad \tau = 1/mB \tag{8}$$

is the relaxation time. That is, the relaxation of the velocity to equilibrium is exponential.

Quiz Problem 9. Describe the physical processes leading to percolation phenomena. Show that for a random graph the order parameter exponent for percolation is $\beta_{mf} = 1$. Given this, and the values $\nu = 1/2, \eta = 0$, show that the Lifshitz argument indicates that for percolation $d_{uc} = 6$.

Solution The order parameter for the percolation problem is the probability that a bond is part of the giant cluster, P_∞. On a random graph which provides a mean field theory for the problem, the infinite cluster probability is found from the equation,

$$P_\infty = 1 - e^{-cP_\infty} \tag{9}$$

Near the critical threshold the order parameter is small, so we carry out a Taylor expansion to find,

$$P_\infty \approx cP_\infty - \frac{1}{2}c^2(P_\infty)^2, \tag{10}$$

From this we find that the percolation threshold is at $c^* = 1$, and that near the percolation threshold the order parameter appoaches zero linearly,

$$P_\infty \sim (c - c^*); \quad \text{so that} \quad \beta = 1 \tag{11}$$

The Lifshitz argument considers the ratio,

$$\frac{C(\xi)}{P_\infty^2} \sim \frac{t^{(d-2)\nu}}{t^{2\beta}} \tag{12}$$

where in this problem $t = c - c^*$. We then consider,

$$(d_{uc} - 2)\nu - 2\beta = 0; \quad \text{with} \quad \beta = 1, \nu = 1/2 \tag{13}$$

which gives $d_{uc} = 6$.

Assigned problems

Assigned Problem 1. Starting from the Ginzburg-Landau free energy (Eq. (9)) of the notes, derive the Ginzburg-Landau equation (10).

Solution Problem 1. Within Ginzburg-Landau theory, the Gibbs' free energy for a superconductor is given by,

$$g_{GL} = \int d^3r \left(\frac{1}{2m}|(i\hbar\nabla + qA)\psi(\vec{r}))|^2 + a(T)|\psi(\vec{r})|^2 + \frac{b(T)}{2}|\psi(\vec{r})|^4 + \frac{B^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - B\cdot H \right). \tag{14}$$

Expanding this expression(and dropping function arguments), we have,

$$g_{GL} = \int d^3r \left(\frac{1}{2m}[(-i\hbar\nabla + qA)\psi^* \cdot (i\hbar\nabla + qA)\psi] + a\psi^*\psi + \frac{b}{2}(\psi^*)^2\psi^2 + \frac{(\nabla\wedge A)^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - (\nabla\wedge A)\cdot H \right). \tag{15}$$

Expanding the first term of (2) yields,

$$g_{GL} = \int d^3r \left(\frac{1}{2m} \sum_{\alpha=1}^{3} [\hbar^2 \frac{\partial \psi^*}{\partial x_\alpha} \frac{\partial \psi}{\partial x_\alpha} + i\hbar q A_\alpha (\psi^* \frac{\partial \psi}{\partial x_\alpha} - \psi \frac{\partial \psi^*}{\partial x_\alpha}) + q^2 A_\alpha^2 \psi^* \psi] \right.$$

$$\left. + \quad a\psi^*\psi + \frac{b}{2}(\psi^*)^2\psi^2 + \frac{(\nabla \wedge A)^2}{2\mu_0} + \frac{\mu_0 H^2}{2} - (\nabla \wedge A) \cdot H \right) \tag{16}$$

Using $\vec{r} = (x_1, x_2, x_3)$, the Euler-Lagrange equation for a variation with respect to ψ^* is,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{\partial g_{GL}}{\partial \psi^*} - \sum_\alpha \frac{\partial}{\partial x_\alpha} (\frac{\partial g_{GL}}{\partial (\frac{\partial \psi^*}{\partial x_\alpha})}) \tag{17}$$

Carring out this variation using equation (3) yields,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} \sum_\alpha (i\hbar q A_\alpha \frac{\partial \psi}{\partial x_\alpha} + q^2 A_\alpha^2 \psi) + a\psi + b|\psi|^2\psi - \frac{1}{2m} \sum_\alpha \frac{\partial}{\partial x_\alpha} (\hbar^2 \frac{\partial \psi}{\partial x_\alpha} - i\hbar q A_\alpha \psi). \tag{18}$$

This is equivalent to,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} (i\hbar \nabla + qA)^2 \psi + a\psi + b|\psi|^2\psi \tag{19}$$

which is the GL equation in an applied field, as required. Another useful form is found by using the London gauge where $\nabla \cdot A = 0$ to find,

$$\frac{\delta g_{GL}}{\delta \psi^*} = \frac{1}{2m} \sum_\alpha (-\hbar^2 \frac{\partial^2 \psi}{\partial x_\alpha^2} + 2i\hbar q A_\alpha \frac{\partial \psi}{\partial x_\alpha} + q^2 A_\alpha^2 \psi) + a\psi + b|\psi|^2\psi = 0 \tag{20}$$

Assigned Problem 2. Starting from Eq. (9) derive Eq. (11) of the notes.
The Euler-Lagrange equation for a variation with respect to one component A_i of the vector potential is,

$$\frac{\delta g_{GL}}{\delta A_i} = \frac{\partial g_{GL}}{\partial A_i} - \sum_\alpha \frac{\partial}{\partial x_\alpha} (\frac{\partial g_{GL}}{\partial (\frac{\partial A_i}{\partial x_\alpha})}) \tag{21}$$

Using Eq. (3), we have,

$$\frac{\delta g_{GL}}{\delta A_i} = i\frac{\hbar q}{2m} (\psi^* \frac{\partial \psi}{\partial x_i} - \psi \frac{\partial \psi^*}{\partial x_i}) + \frac{q^2}{m} A_i \psi^* \psi - \sum_\alpha \frac{1}{2\mu_0} \frac{\partial}{\partial x_\alpha} (\frac{\partial (\nabla \wedge A)^2}{\partial (\frac{\partial A_i}{\partial x_\alpha})}) \tag{22}$$

The term $B \cdot H$ does not contribute as it is linear in the derivatives of A and hence is zero after the application of the second term in the Euler-Lagrange equation. The expansion of $(\nabla \wedge A)^2$ is,

$$(\nabla \wedge A) \cdot (\nabla \wedge A) = (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3})^2 + (\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1})^2 + (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2})^2 \tag{23}$$

Using this expression to evaluate the last term of Eq. (9) in the case $A_i = A_1$, yields

$$\sum_\alpha \frac{\partial}{\partial x_\alpha} (\frac{\partial (\nabla \wedge A)^2}{\partial (\frac{\partial A_1}{\partial x_\alpha})}) = 2(-\frac{\partial^2 A_2}{\partial x_1 \partial x_2} + \frac{\partial^2 A_1}{\partial x_2^2} + \frac{\partial^2 A_1}{\partial x_3^2} - \frac{\partial^2 A_3}{\partial x_1 \partial x_3}) \tag{24}$$

Similar expressions hold for A_2 and A_3. Now note that $\nabla \wedge (\nabla \wedge A) = \nabla(\nabla \cdot A) - \nabla^2 A$. The first element of this expression is,

$$(\nabla(\nabla \cdot A) - \nabla^2 A)_1 = \frac{\partial^2 A_1}{\partial x_1^2} + \frac{\partial^2 A_2}{\partial x_1 x_2} + \frac{\partial^2 A_3}{\partial x_1 x_3} - \frac{\partial^2 A_1}{\partial x_1^2} - \frac{\partial^2 A_1}{\partial x_2^2} - \frac{\partial^2 A_1}{\partial x_3^2}$$

We thus find the identity,

$$\frac{\delta(\nabla \wedge A)^2}{\delta A} = 2\nabla \wedge (\nabla \wedge A) = 2[\nabla(\nabla \cdot A) - \nabla^2 A] \tag{25}$$

We shall use this identity in Problem 3 below.
We also have,

$$\nabla \wedge (\nabla \wedge A) = \nabla \wedge B = \mu_0 J \tag{26}$$

Substituting these results into Eq. (11) yields the simple result,

$$-\sum_\alpha \frac{1}{2\mu_0}\frac{\partial}{\partial x_\alpha}(\frac{\partial(\nabla \wedge A)^2}{\partial(\frac{\partial A_i}{\partial x_\alpha})}) = j$$

Using this in Eq. (9) we finally get,

$$j_s = -i\frac{\hbar q}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q^2}{m}A_i\psi^*\psi \tag{27}$$

which is Eq. (11) of the notes, as required.

Assigned Problem 3. By doing a variation of the Helmholtz free energy (15) prove Eq. (17).

Solution. We apply the Euler-Lagrange equations to,

$$f_{London} = \frac{1}{2\mu_0}\int dV[\lambda^2(\nabla \wedge B)^2 + B^2]. \tag{28}$$

Using the identity (25) above for the case of a variation with respect to B, and using $\nabla \cdot B = 0$, yields Eq. (15) of the notes.

Assigned Problem 4. Using London theory, find the Gibb's free energy per unit length of a vortex in a superconductor in an external field H.
Solution. The *energy*(per unit length) of an isolated vortex is given by,

$$\epsilon_1 = \int_0^\infty (\frac{B^2}{2\mu_0} + \frac{1}{2}\rho_s v_s^2)2\pi r dr \tag{29}$$

The first term is the field energy and the second is the kinetic energy of the superconducting electrons (see Eq. (4)). The supercurrent is related to the velocity by,

$$J = n_b q v_s \tag{30}$$

and using Maxwells equation, we then write,

$$\epsilon_1 = \frac{1}{2\mu_0}\int_0^\infty (B^2 + \lambda^2\mu_0^2 J^2)2\pi r dr \tag{31}$$

which explicitly shows the contributions of the field and the current. In the large λ/ξ limit, this is dominated by the regime $\xi \le r \le \lambda$, so we find an approximate value of the vortex energy by using,

$$\epsilon_1 \approx \frac{1}{2\mu_0}(\frac{\phi_0}{2\pi\lambda^2})^2 2\pi \int_\xi^\lambda [r(Ln(\frac{\lambda}{r}))^2 + r(\frac{\lambda}{r})^2]dr$$

$$= \frac{\phi_0^2}{4\pi\mu_0\lambda^2}\int_{\xi/\lambda}^1 [x(Ln(x))^2 + \frac{1}{x}]dx$$

$$= \frac{\phi_0^2}{4\pi\mu_0\lambda^2}[\frac{x^2}{2}(\frac{1}{2} - Ln(x) + Ln(x)^2) + Ln(x)]|_{\frac{\xi}{\lambda}}^{1}$$

$$\approx \frac{\phi_0^2}{4\pi\mu_0\lambda^2}[Ln(\frac{\lambda}{\xi})] \tag{32}$$

Notice that the energy cost of forming the vortex is dominated by the kinetic energy of the superconducting electrons (the logarithmic term). The energy cost due to the magnetic field is relatively small.

Assigned Problem 5. Using the Gibb's free energy within London theory, demonstrate that the triangular lattice vortex array has lower energy than the square lattice array, for a fixed applied field, $H > H_{c1}$, which is close to H_{c1}.

Solution
The energy to add the N^{th} vortex to a vortex lattice is given by the energy of the added vortex plus the vortex-vortex interactions. Within London theory, the Gibbs free energy (or chemical potential) to add the N^{th} vortex to a vortex array is given by,

$$\delta g = (\epsilon_1 - \phi_0 H) + \sum_j \frac{B_0\phi_0}{\mu_0} K_0(|\vec{r} - \vec{r}_j|/\lambda) \tag{33}$$

Using $H_{c1} = \epsilon_1/\phi_0$, the relation between the applied field and the lattice spacing is found by setting $\delta g = 0$, which yields,

$$H - H_{c1} = \frac{B_0}{\mu_0 N} \sum_{ij} K_0(|\vec{r}_i - \vec{r}_j|/\lambda) \tag{34}$$

This fixes the lattice spacing, a, of a flux array. This spacing is different for different lattice structures. Associated with the lattice spacing there is a total number of vortices inside the superconductor. This total number of vortices, at fixed a, is larger for the triangular lattice, N_{tr} than it is for the square lattice, $N_{tr} > N_{sq}$. In fact the highest packing possible is for the triangular lattice case.
In setting up a vortex lattice, the gain in Gibb's free energy is given by,

$$\delta G = N(\epsilon_1 - \phi_0 H) + \sum_{ij} \frac{B_0\phi_0}{2\mu_0} K_0(|\vec{r}_i - \vec{r}_j|/\lambda) \tag{35}$$

Notice that there is a factor of two in the second term(compared to Eq. (33)), which is the energy cost of vortex-vortex repulsion, due to the fact that the energy cost in setting up the flux lattice is small for the first vortices added to the lattice. Using Eq. (34), this can be rewritten as,

$$\phi_0 \frac{\delta G}{N} = (H_{c1} - H) + \frac{1}{2}(H - H_{c1}); \quad \text{so that} \quad \frac{\delta G}{N} = \frac{1}{2\phi_0}(H_{c1} - H) \tag{36}$$

This is the energy gain per vortex in setting up the lowest energy vortex lattice. Notice that if we consider only nearest neighbor interactions, the energy per vortex is *the same* no matter what the vortex lattice structure. However the *number* of vortices which can be added to the superconductor does depend on the structure of the final vortex lattice. Since the number of vortices in the case of a square lattice is smaller than the number added for the triangular lattice, the total energy is lowest for the case of a triangular lattice. Notice that this result is general, the triangular lattice is the true ground state because it has the highest possible vortex density.

Assigned Problem 6. Prove that $\chi \sim \int dVC(r)$ where for a lattice the integral over volume is replaced by a sum over lattice sites.

Solution The correlation function (C_{ij}) and Susceptibility χ are given by,

$$C_{ij} =< S_i S_j > - < S_i >< S_j >; \quad k_B T N\chi =< M^2 > - < M >^2; \quad where, \quad M =< \sum_i S_i > \tag{37}$$

Expanding the expression for χ we have,

$$k_BTN\chi =< (\sum_i S_i)^2 > - < \sum_i S_i >^2 = \sum_{ij} [< S_iS_j > - < S_i >< S_j >] = \sum_{ij} C_{ij} \to V \int d^d r C(r) \tag{38}$$

Solution

Assigned Problem 7. Show that the RG equation for the three dimensional nearest neighbor, ferromagnetic Ising model on a simple cubic lattice, within the Migdal Kadanoff scheme is,

$$Tanh(K') = Tanh^2(3K) \tag{39}$$

Find the fixed points of this RG equation. By carrying out a linear expansion near the fixed point, find the correlation length exponent within this approximation. Is it close to the correct value? Sketch the RG flow for this system.

Solution The only change to the one dimensional solution is that the renormalization group equation is that there are now two extra bonds moved to each remaining bond, so that,

$$tanh(K') = tanh^2(3K) \tag{40}$$

This equation has three fixed points $K^* = 0, 1, K^* = .1203$. Due to the scaling form of the free energy, the behavior of the RG equations near the fixed point are,

$$\delta K'(K) = b^{D_t}|K^* - K|; \qquad \text{or} \qquad v'(v) = b^{D_t}|v^* - v| \tag{41}$$

where $v = tanh(K)$ and in our decimation procedure $b = 2$. We use a leading order Taylor expansions about K_*,

$$K^* + \delta K'(K) = ArcTanh(Tanh^2(dK^*)) + (K - K^*)\frac{2dTanh(dK^*)}{1 + Tanh^2(dK^*)} \tag{42}$$

so that,

$$b^{D_t} = \frac{2dTanh(dK^*)}{1 + Tanh^2(dK^*)} \tag{43}$$

and hence

$$D_t = ln\left(\frac{2dTanh(dK^*)}{1 + Tanh^2(dK^*)}\right)/ln(2) = 0.891; \qquad so\ that, \qquad \nu = 1/D_t = 1.12 \tag{44}$$

The value of ν in three dimensions is 0.63, so the simple Migdal-Kadanoff method used here is not very good in three dimensions.

Note that the equation above, with $d = 2$, along with $K^* = 0.305$ gives $D_t = 0.747$ as found in lectures for the square lattice.

Assigned Problem 8.

(i) In reduced units ($P_r = P/P_c, T_r = T/T_c, v_r = v/v_c$), the van der Waals equation of state becomes,

$$P_r = \frac{8T_r}{3v_r - 1} - \frac{3}{v_r^2} \tag{45}$$

Show that the spinodal lines of the model are given by,

$$P_r = \frac{3}{v_r^2} - \frac{2}{v_r^3} \tag{46}$$

(ii) By using the condition when $E_{barrier} = 0$, find an expression for the spinodal line of the Ising model on a square lattice, within the droplet theory.

Solution

(i) The spinodal lines are found using $\partial P_r/\partial v_r = 0$, so that,

$$-\frac{24T_r}{(3v_r-1)^2}+\frac{6}{v_r^2}=0; \quad \text{so that} \quad T_r=(3v_r-1)^2/(4v_r^3) \tag{47}$$

substitution of this expression for T_r into the reduced van der Waals equation (Eq. (45)) gives the spinodal equation, as required.

(ii) The energy barrier is given by,

$$E_{barrier}=E_{droplet}(R_c)=-2hc_d(\frac{(d-1)k_d\sigma}{dc_dh})^d+2\sigma k_d(\frac{(d-1)\sigma k_d}{dc_dh})^{d-1} \tag{48}$$

We write this in the from,

$$E_{barrier}=2h(\frac{\sigma}{h})^d[k_d(\frac{(d-1)k_d}{dc_d})^{d-1}-c_d(\frac{(d-1)k_d}{dc_d})^d]=A_bh(\frac{\sigma}{h})^d \tag{49}$$

where A_b is a constant. The spinodal line is approximated by setting $k_BT=E_{barrier}$, so that,

$$k_BT_s\approx A_bh_s(\frac{\sigma}{h_s})^d; \quad \text{so that} \quad h_s\approx\frac{1}{(k_BT_s/A_b\sigma^d)^{1/(d-1)}} \tag{50}$$

Assigned Problem 9.

(i) Prove the result (107) of notes.

(ii) Fill in the details of the derivation of Eq. (113) from (110).

Solution

(i) We start with the Langevin equation,

$$m\frac{d\vec{v}}{dt}=-\frac{\vec{v}}{B}+\eta'(t) \tag{51}$$

where B is the damping coefficient and $\eta'(t)$ is the random force, where $<\eta'(t)>=0$ and B is the damping coefficient. We take a time average of the Langevin equation which leads to,

$$m\frac{d<\vec{v}>}{dt}=-\frac{<\vec{v}>}{B}; \quad \text{so that} \quad \vec{v}(t)>=\vec{v}(0)e^{-t/\tau} \quad \text{where} \quad \tau=1/mB(52)$$

(ii) We find the mean square distance travelled by the particle by taking a dot product of $\vec{r}$ with the Langevin equation and then averaging over the noise. To carry this out, we use the results,

$$\vec{r}\cdot\vec{v}=\frac{1}{2}\frac{d(\vec{r}\cdot\vec{r})}{dt}; \quad \vec{r}\cdot\frac{d\vec{v}}{dt}=\frac{1}{2}\frac{d^2(\vec{r}\cdot\vec{r})}{dt^2}-v^2; \quad <\vec{r}\cdot\eta(t)>=0.0 \tag{53}$$

to find that,

$$\frac{d^2<r^2>}{dt^2}+\frac{1}{\tau}\frac{d<r^2>}{dt}=2<v^2>=\frac{dk_BT}{m} \tag{54}$$

where the last expression comes from using the equipartition theorem for a monatomic particle system. Solving this equation yields,

$$<r^2>=\frac{2dk_BT}{M}\tau^2[\frac{t}{\tau}-(1-e^{-t/\tau})] \quad \text{so as } t\to\infty \quad <r^2>=2dBk_BTt \tag{55}$$

Comparing with the diffusion equation result yields $B=D/k_BT$, which is the Einstein relation for brownian motion. It is one example of a fluctuation-dissipation result where the close to equilibrium response (the damping) is related to the dynamical fluctuations at equilibrium (the diffusion). The damping coefficient B can be measured as the "mobility" defined as the ratio of the terminal velocity over the driving force.